$$1+1+1+\ldots+1=a+b+c$$
$$\omega+1+\omega+\omega+1+\ldots=a\omega^2+b\omega+c=a\overline{\omega}+b\omega+c$$
$$\overline{\omega}+1+\overline{\omega}+\overline{\omega}+1+\ldots=a\overline{\omega}^2+b\overline{\omega}+c=a\omega+b\overline{\omega}+c$$

The exponents in P have the form $(2n+1)^2$ for n from 0 to 2022, resulting in 2023 terms. The exponent is a multiple of 3 precisely if $n \equiv 1$ mod 3, so the second and third equations have 674 terms of 1 and 1349 terms that are not 1. We rewrite our equations as

$$2023=a+b+c \quad (1)$$
$$674+1349\omega=a\overline{\omega}+b\omega+c \quad (2)$$
$$674+1349\overline{\omega}=a\omega+b\overline{\omega}+c \quad (3)$$

Adding equations 2 and 3 results in $-1=-a-b+2c$. Adding this to equation 1 yields $2022=3c$, or $c=674$. Subtracting equation 3 from equation 2 yields $1349i\sqrt{3}=-ai\sqrt{3}+bi\sqrt{3}$, or $1349=-a+b$. This can be solved with equation 1 as a two-variable system to obtain $a=0$ and $b=1349$. So, the requested remainder is $1349x+674$.

Also solved by Troy Williamson, Texas State Technical College, Abilene, TX.

It All Adds Up!

K-2 Proposed by Michael W. Ecker, inspired by QK-1 in that same issue. An additive sequence $< a_n >$ has the property that $a_n = a_{n-1} + a_{n-2}\ \forall n \geq 3$, with given initial $a_1 = a$ and $a_2 = b$.

a) Find a formula for $\sum_{i=1}^{n} a_i$ in terms only of *a*, *b*, and other known elements.

b) Show that this sum equals $a_{n+2}-1$ (as with the Fibonacci sequence) if and only if $b=1$.

c) Show that this sum equals a_{n+2} (as with multiples of the Fibonacci sequence) if and only if $b=0$. What is a_{n+2} then in this case?

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY. By the defining relation,

$$\sum_{i=1}^{n} a_i = a+b+\sum_{i=3}^{n} a_i = a+b+\sum_{i=3}^{n}(a_{i+2}-a_{i+1})$$
$$= a+b+(a_{n+2}-a_4) = a+b+a_{n+2}-(a+2b) = a_{n+2}-b,$$

and this formula immediately implies the initial statements of parts b) and c). If $b = 0$, then $a_{n+2} = F_n \cdot a$, where F_n denotes the *n*th Fibonacci number. (Problems Editor's Note: This is a result easily proved by induction – or as a consequence of the more general relation $a_n = F_{n-2}a + F_{n-1}b$ that has previously appeared more than once in this journal, including in this column.)

Also solved by Raymond N. Greenwell (Emeritus), Hofstra University, Hempstead, NY; and the proposer.

All in the Family

K-3 Proposed by Stephen L. Plett. Choose any real number $p>1$ and form a family of curves $F:\left\{y_K = Kx^{(p^2)} \mid x, K\ real\right\}$. a) Show that the orthogonal trajectories constitute a family of ellipses, and b) express their common eccentricity in terms of *p*.

Solution by Raymond N. Greenwell. a) For the given family of curves, the slopes are given by $\frac{dy}{dx} = K(p^2)x^{p^2-1} = \frac{y}{x^{p^2}} \cdot p^2 x^{p^2-1} = \frac{p^2 y}{x}$. Therefore, the orthogonal trajectories are given by the differential equation $\frac{dy}{dx} = -\frac{x}{p^2 y}$. Separating variables and integrating gives $\int p^2 y\, dy = -\int x\, dx$ and $\frac{p^2y^2}{2} = -\frac{x^2}{2} + C'$ or $x^2 + \frac{y^2}{1/p^2} = C$, where $C = 2C'$. This is the equation of an ellipse with $a^2 = 1$ and $b^2 = \frac{1}{p^2}$, where $b^2 < 1$ since $p > 1$.

b) For ellipses of the form in part a), the eccentricity is $e = \frac{\sqrt{a^2-b^2}}{a} = \frac{\sqrt{1-1/p^2}}{1} = \frac{\sqrt{p^2-1}}{p}$.

Also solved by Ivan Retamoso; Bryan Wilson; and the proposer.

Keeping It Real!

K-4 Proposed by Michael W. Ecker. With complex variable $z = x + yi$ and complex constant $c = a + bi$, let $w = (z-c)^2$. Identify and graph all points (x, y, w) for which *w* is real-valued.

Similar Solutions by Raymond N. Greenwell, Ivan Retamoso, Albert Natian, and the proposer (each independently).

To visualize this, think of the *x*, *y* axes as making up a "flat" base plane, and the third dimension "up" will be Re(*w*) (real part of *w*). As $(p+qi)^2 = p^2 - q^2 + 2pqi$ is real iff $p=0$ or $q=0$, so $w=(z-c)^2 = ((x-a)+(y-b)i)^2$ is real iff $x=a$ or $y=b$.

Case A: $x=a$ describes a plane parallel to the *y*, Re(*w*) plane. Substitute $x=a$ into *w* to get $w=-(y-b)^2$. So, one portion of the solution is the "max" parabola $w=-(y-b)^2$ in the plane $x=a$. Note that the vertex point (a,b) in the *x*, *y* plane is the sole point on this part of the graph with $w=0$.

Case B: $y = b$ describes a plane parallel to the x, Re(w) plane. Substitute $y = b$ into w to get $w = (x-a)^2$. So, one portion of the solution is the "min" parabola $w = (x-a)^2$ in the plane $y = b$. Note that the vertex point (a,b) in the x, y plane is the sole point on this part of the graph with $w = 0$.

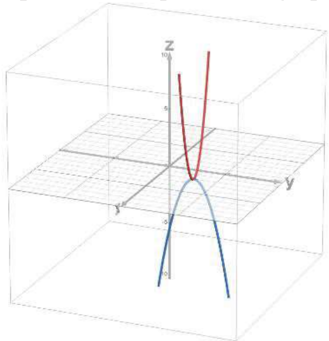

Figure with third dimension up actually being Re(w)

Trouble Keeping Time

K-5 Proposed by Bryan Wilson. A poorly designed clock has hour and minute hands the same length and shape. Assume the hands move continuously. There are some times during the day during which, even with precise measurement, there are two possible interpretations of the time (such as about 1:21 or 4:07). How many such times between noon and midnight will there be two possible interpretations of the time?

Similar solutions by Raymond N Greenwell and the Proposer. Let x be the position of the hour hand in minutes after noon inside the interval [0, 720). Similarly, let y be the position of the minute hand, and note that $y \equiv 12x \bmod 720$. A position is ambiguous if we also have $x \equiv 12y \bmod 720$ since we use the same congruence but reversing the roles of the hands. Substituting the first congruence into the second, we obtain $x \equiv 12(12x) \bmod 720$ which can be rewritten $143x \equiv 0 \bmod 720$. Thus $143x = 720k$ or $x = \frac{720}{143}k$ for some integer k. There are 143 values of k that result in unique values of x mod 720. These come every $\frac{720}{143} \approx 5.035$ minutes starting at noon. However, 11 of these correspond to times that the hour and minute hands are in the same position, which is not ambiguous. So, the total number of ambiguous times is 132.

Also solved by Troy Williamson.

Rationally Sequential Is Sequentially Rational

K-6 Proposed by Michael W. Ecker. a) Given three distinct rational numbers, prove there exists an arithmetic sequence that includes them. b) Is the sequence unique? If there is more than one such arithmetic sequence, how are they related? c) Same questions for any finite number $n > 1$ of rationals.

Essentially Same Solution by the Proposer and Raymond N. Greenwell (independently).

a) We have $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$, with each fraction in lowest terms. Let $L = bdf$. Then we can write each of the three fractions in the form $\frac{k}{L}$ where k runs through the integers. Specifically,

$$\frac{a}{b} = \frac{a}{b} \cdot \frac{df}{df} = \frac{adf}{L}, \quad \frac{c}{d} = \frac{c}{d} \cdot \frac{bf}{bf} = \frac{bcf}{L}, \quad \frac{e}{f} = \frac{e}{f} \cdot \frac{bd}{bd} = \frac{bde}{L}.$$

However, the set of these fractions $\frac{k}{L}$ constitutes an arithmetic sequence with common difference $\frac{1}{L}$. b) The sequence is not unique. For example, the sequence that has the greatest possible common difference is the one for which we repeat step a) but with L instead equal to the least common multiple of b, d, f. (In case these three integers b, d, f are pairwise relatively prime, L is the same as before.) In fact, *all* arithmetic sequences containing the given three rational numbers have a common difference that is an integral divisor of this particular $\frac{1}{L}$, where L is the least common multiple of b, d, f. In other words, this particular arithmetic sequence, for this one value of L, is the "fattest" one – meaning, it has the greatest possible common difference. c) This argument generalizes to any number n of rationals. Just let L first be the product of the n denominators to get one representation, and then let L instead be the least common multiple of the n denominators, doing so in the same way as we just did for 3 denominators.

Also solved by Yutong Liu, East Los Angeles College, Los Angeles, CA; and Troy Williamson.

Michael W. Ecker had a 45-year career as a mathematics professor, most of it at Pennsylvania State University's Wilkes-Barre campus. He retired from teaching in 2016. His PhD in mathematics was from the City University of New York (1978). Published 500+ times as a mathematician or computer journalist, Mike also served on national committees responsible for creating competitive national exams, and was the Founding Problem Section Editor of *The AMATYC Review* (1981-1997). As a recreational mathematician, he published his own newsletter, *Recreational & Educational Computing* (1986-2007). For free PDF copies of **REC**, visit https://dr-michael-ecker.weebly.com.