$d \in S$. Hence $d = x_1$, so $x_2 = 2x_1$. Next consider $d' = x_3 - x_2 < x_3$. This is in S and forces either $d = x_1$ or $d' = x_2$. In the latter case, we would have $x_3 = x_2 + d' = 2x_2 = 4x_1$. This in turn implies $x_2 = 2x_1 < 3x_1 < 4x_1 = x_3$, which is impossible, because $x_3 - x_1 = 3x_1$ would have to be an element of S by the definition, and yet there can be no element between x_2 and x_3. Similar arguments then show that each difference is the same $d = x_1$. I'll omit the details. Bottom line is that we may think of every such possible finite set S as being of the form $x_1 S = x_1\{1,2,3,...n\} = \{x_1, 2x_1, 3x_1, ..., nx_1\}$, $x_1 \neq 0$.

QN-2 If set S has a largest element m, then for all elements $x < m$, we wish to show that $f(x) = f(m)$. (This could be for a finite set or a closed interval, for instance.) As in QN-1, by (*) we can pair each element x in S with another element $m - x$ in S, so applying (**) to $m - x < m$ we get $f(m-(m-x)) = f(x) = f(m)$. So, f is constant in this case, its value always being $f(m)$. Now suppose S lacks a largest element (such as with an open interval, the set of all natural numbers, etc.). Choose any two elements $a, b \in S$ with $a < b$. Now choose a still larger element c. Once again, we can work with $c - a$ and $c - b$ both in S to get $f(c-(c-a)) = f(a) = f(c)$ and likewise get $f(b) = f(c)$. Hence, $f(b) = f(a)$ for any pair a, b in S. We do obtain that f must be a constant function in any case.

QN-3 We let $(1,1)$ be the center of circle A and observe that the center of circles S and L is the origin. Let O represent the origin, P the point where S is tangent to A, and R the point where A is tangent to L. The radius of L is $OA + AR = \sqrt{2}+1$, while the radius of S is $OA - PA = \sqrt{2}-1$. The desired ratio is now $\frac{\sqrt{2}+1}{\sqrt{2}-1} = \left(\sqrt{2}+1\right)^2 = 3+2\sqrt{2}$.

QN-4 We try a "greedy' approach with the same numbers of each coin. If we use q quarters, then we also use q each of dimes, nickels, and pennies, giving $.25q + .10q + .05q + .01q = .41q = 1000$ as the total value. Thus, we would get $q = \frac{1000}{.41} \approx 2439.0244$ so we use 2,439 each of quarters, dimes, and nickels giving a total of \$975.60, leaving 100000-97560 = 2,440 pennies. So, 9,757 coins is the total minimum number.

QN-5 Because f is even, the area we seek over the entire real line is the same as the area under the curve $F(x) = 2f(x) = \frac{1}{\lfloor x \rfloor !}$ over $[0,\infty)$. Break this interval up using integers, as follows: On [0, 1), $F(x) = \frac{1}{\lfloor x \rfloor !} = \frac{1}{0!} = 1$; on [1, 2), $F(x) = \frac{1}{\lfloor x \rfloor !} = \frac{1}{1!} = 1$; on [2, 3), $F(x) = \frac{1}{\lfloor x \rfloor !} = \frac{1}{2!} = \frac{1}{2}$; and on each interval $[n, n+1)$, $F(x) = \frac{1}{\lfloor x \rfloor !} = \frac{1}{n!}$. The total area is then the sum of these areas, or $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.

QN-6 Solution I: From the definition of BA, we have the inequality $\frac{200x + 300(162-x)}{162} \geq 240$. Cross-multiply to get $300(162) - 100x \geq 240 \cdot 162$ or $60(162) \geq 100x$ yielding $x \leq 97.2$ games. As we want the *largest possible value of x with BA at least 240*, we find $x = 97$ games. Easier Solution II: As 240 is 2/5 of the way from 200 to 300, we carve up the 162-game span into fifths, recognizing that we need the point that is just 2/5 of the way from 162 down to 0, or 3/5 of the way from 0 up to 162. So, $x = 0 + \frac{3}{5} \cdot 162 = 97.2$ games. Once again, we have to truncate this to 97.

Solutions to Problems from Previous Issues

What Remains Within

K-1 Proposed By Mark Moodie, San Jacinto College, North Campus, Houston, TX. Find the remainder when $x + x^9 + x^{25} + x^{49} + \cdots + x^{4045^2}$ is divided by $x^3 - 1$.

Solution by Ivan Retamoso, Borough of Manhattan Community College, New York, NY, and Bryan Wilson, Contributing Editor (independently). Label the cube roots of unity by 1, $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \overline{\omega}$. Note that $\overline{\omega}^2 = \omega$, $\omega + \overline{\omega} = -1$, and $\omega - \overline{\omega} = i\sqrt{3}$. Set $P(x) = x + x^9 + x^{25} + x^{49} + ... + x^{4045^2}$ $= Q(x)(x^3-1) + ax^2 + bx + c$. Substituting each of the cube roots of unity into this identity and noting that $1^3 = \omega^3 = (\overline{\omega})^3 = 1$ results in the three equations

$$1+1+1+\ldots+1=a+b+c$$
$$\omega+1+\omega+\omega+1+\ldots=a\omega^2+b\omega+c=a\bar{\omega}+b\omega+c$$
$$\bar{\omega}+1+\bar{\omega}+\bar{\omega}+1+\ldots=a\bar{\omega}^2+b\bar{\omega}+c=a\omega+b\bar{\omega}+c$$

The exponents in P have the form $(2n+1)^2$ for n from 0 to 2022, resulting in 2023 terms. The exponent is a multiple of 3 precisely if $n \equiv 1$ mod 3, so the second and third equations have 674 terms of 1 and 1349 terms that are not 1. We rewrite our equations as

$$2023=a+b+c \qquad (1)$$
$$674+1349\omega=a\bar{\omega}+b\omega+c \qquad (2)$$
$$674+1349\bar{\omega}=a\omega+b\bar{\omega}+c \qquad (3)$$

Adding equations 2 and 3 results in $-1=-a-b+2c$. Adding this to equation 1 yields $2022=3c$, or $c=674$. Subtracting equation 3 from equation 2 yields $1349i\sqrt{3}=-ai\sqrt{3}+bi\sqrt{3}$, or $1349=-a+b$. This can be solved with equation 1 as a two-variable system to obtain $a=0$ and $b=1349$. So, the requested remainder is $1349x+674$.

Also solved by Troy Williamson, Texas State Technical College, Abilene, TX.

It All Adds Up!

K-2 Proposed by Michael W. Ecker, inspired by QK-1 in that same issue. An additive sequence $<a_n>$ has the property that $a_n=a_{n-1}+a_{n-2}\ \forall n\geq 3$, with given initial $a_1=a$ and $a_2=b$.

a) Find a formula for $\sum_{i=1}^{n} a_i$ in terms only of a, b, and other known elements.
b) Show that this sum equals $a_{n+2}-1$ (as with the Fibonacci sequence) if and only if $b=1$.
c) Show that this sum equals a_{n+2} (as with multiples of the Fibonacci sequence) if and only if $b=0$. What is a_{n+2} then in this case?

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY. By the defining relation,

$$\sum_{i=1}^{n} a_i = a+b+\sum_{i=3}^{n} a_i = a+b+\sum_{i=3}^{n}(a_{i+2}-a_{i+1})$$
$$=a+b+(a_{n+2}-a_4)=a+b+a_{n+2}-(a+2b)=a_{n+2}-b,$$

and this formula immediately implies the initial statements of parts b) and c). If b = 0, then a_{n+2} = $F_n \cdot a$, where F_n denotes the nth Fibonacci number. (Problems Editor's Note: This is a result easily proved by induction – or as a consequence of the more general relation $a_n=F_{n-2}a+F_{n-1}b$ that has previously appeared more than once in this journal, including in this column.)

Also solved by Raymond N. Greenwell (Emeritus), Hofstra University, Hempstead, NY; and the proposer.

All in the Family

K-3 Proposed by Stephen L. Plett. Choose any real number $p>1$ and form a family of curves $F:\left\{y_K=Kx^{(p^2)} \mid x, K\ real\right\}$. a) Show that the orthogonal trajectories constitute a family of ellipses, and b) express their common eccentricity in terms of p.

Solution by Raymond N. Greenwell. a) For the given family of curves, the slopes are given by $\frac{dy}{dx}=K(p^2)x^{p^2-1}=\frac{y}{x^{p^2}}\cdot p^2x^{p^2-1}=\frac{p^2y}{x}$. Therefore, the orthogonal trajectories are given by the differential equation $\frac{dy}{dx}=-\frac{x}{p^2y}$. Separating variables and integrating gives $\int p^2y\,dy=-\int x\,dx$ and $\frac{p^2y^2}{2}=-\frac{x^2}{2}+C'$ or $x^2+\frac{y^2}{1/p^2}=C$, where $C=2C'$. This is the equation of an ellipse with $a^2=1$ and $b^2=\frac{1}{p^2}$, where $b^2<1$ since $p>1$.

b) For ellipses of the form in part a), the eccentricity is $e=\frac{\sqrt{a^2-b^2}}{a}=\frac{\sqrt{1-1/p^2}}{1}=\frac{\sqrt{p^2-1}}{p}$.

Also solved by Ivan Retamoso; Bryan Wilson; and the proposer.

Keeping It Real!

K-4 Proposed by Michael W. Ecker. With complex variable $z=x+yi$ and complex constant $c=a+bi$, let $w=(z-c)^2$. Identify and graph all points (x,y,w) for which w is real-valued.

Similar Solutions by Raymond N. Greenwell, Ivan Retamoso, Albert Natian, and the proposer (each independently).

To visualize this, think of the x, y axes as making up a "flat" base plane, and the third dimension "up" will be Re(w) (real part of w). As $(p+qi)^2=p^2-q^2+2pqi$ is real iff $p=0$ or $q=0$, so $w=(z-c)^2=((x-a)+(y-b)i)^2$ is real iff $x=a$ or $y=b$.

Case A: $x=a$ describes a plane parallel to the y, Re(w) plane. Substitute $x=a$ into w to get $w=-(y-b)^2$. So, one portion of the solution is the "max" parabola $w=-(y-b)^2$ in the plane $x=a$. Note that the vertex point (a,b) in the x, y plane is the sole point on this part of the graph with $w=0$.