The Problem Corner

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The Purpose of ***The Problem Corner*** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello, problem solvers!

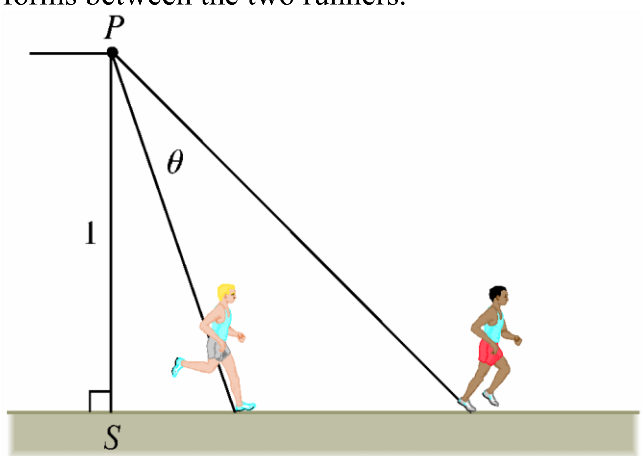
I'm delighted to announce that we have received both precise and thoughtful solutions for Problem 28 and Problem 29 in The Problem Corner. These submissions excel in both accuracy and strategic problem-solving. Our aim is to highlight exemplary solutions that foster inspiration and advance mathematical insights on a global scale.

Solutions to **Problems** from the Previous Issue.

“Running a race” problem.

Problem 28
Proposed by Ivan Retamoso, BMCC, USA.

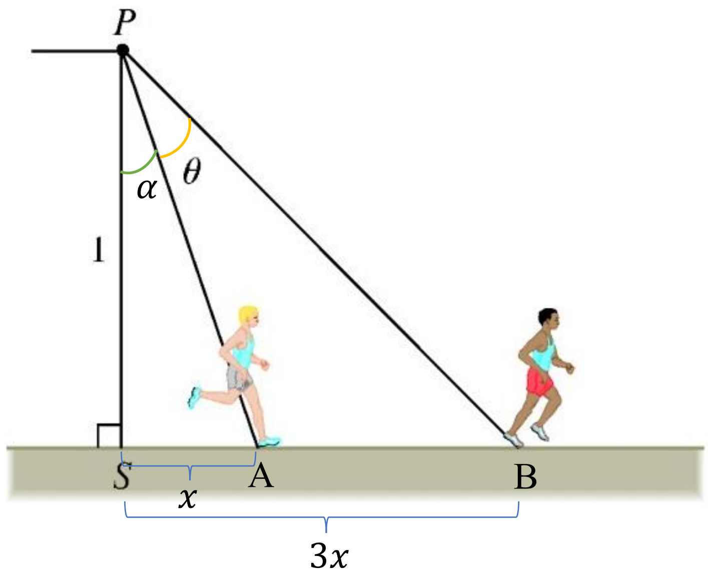
An observer is positioned at point P, one unit away from a track. Two runners begin at point S, which is illustrated in the diagram, and move along the track. One of the runners runs at a speed three times faster than the other. Determine the maximum angle θ that the observer's line of sight forms between the two runners.



Solution to problem 28
By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

After carefully labeling the angles and distances, our solver uses the trigonometric expansion for the tangent of the sum of two angles to express the tangent of an angle in terms of a single variable. Finally, by leveraging the fact that for acute angles, the tangent function and the angle increase together, the function is maximized to find the solution.

We can denote points A, B, angle α and distance covered by runners in terms of x on the figure as shown below.



By using the tan $(a + b)$ sum formula on the right-angled triangle PSB:
$\tan(\alpha + \theta) = 3x$
$\frac{\tan\alpha + tan\theta}{1 - tan\alpha.tan\theta} = 3x$

Knowing that $tan\alpha = x$ in the right-angled triangle PSA, we substitute $tan\alpha = x$ and get:

$\frac{x + tan\theta}{1 - x.tan\theta} = 3x$

After necessary algebraic operations, we get:

$tan\theta = \frac{2x}{1+3x^2}$

Now we need the differentiate this function to get the maximum value of θ.
Notice that for θ an acute angle, $\tan\theta$ is strictly increasing so that θ is maximized when $\tan\theta$ is maximized.

$\left(\frac{2x}{1+3x^2}\right)' = 0$

By using the quotient rule, we get:

$\frac{2-6x^2}{(1+3x^2)^2} = 0$

Then,

$x = \dfrac{\sqrt{3}}{3}$

To be sure that $tan\theta$ gets maximized when $x = \frac{\sqrt{3}}{3}$, you may use a behavior table or the second derivative test.

Now, if we substitute $x = \frac{\sqrt{3}}{3}$,

$$tan\theta = \frac{2\frac{\sqrt{3}}{3}}{1+3\left(\frac{\sqrt{3}}{3}\right)^2} = \frac{\sqrt{3}}{3} \Rightarrow \theta = 30^0 = \frac{\pi}{6}$$

“STOP sign” probability problem.

Problem 29

Proposed by Ivan Retamoso, BMCC, USA.

A regular octagon $ABCDEFGH$ has sides that are 2 units in length. The points W, X, Y, and Z are the midpoints of the sides $\overline{AB}$, $\overline{CD}$, $\overline{EF}$, and $\overline{GH}$, respectively. Find the probability that a point chosen uniformly at random from inside the octagon $ABCDEFGH$ will be located inside the quadrilateral $WXYZ$. Give your answer in exact form.

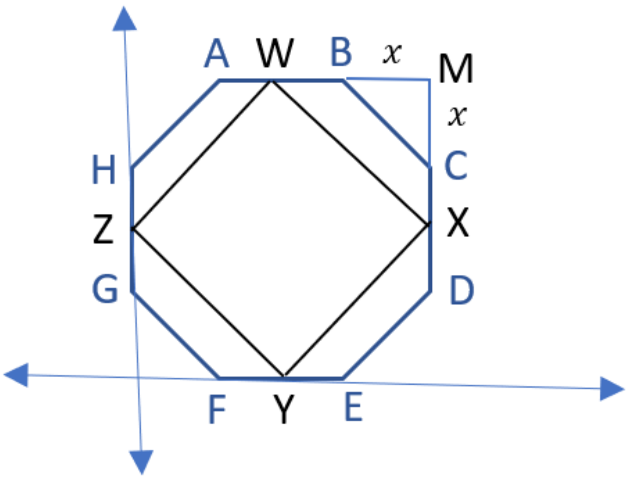
First solution to problem 29

By Dr. Hosseinali Gholami, University Putra Malaysia, Serdang, Malaysia

Our solver presents two solutions: one based on Cartesian geometry, utilizing the distance and slope formulas, and the other based on the Pythagorean theorem and the properties of isosceles triangles. Both solutions are explained in detail for your understanding and enjoyment.

Solution 1:

The following shape shows the regular octagon $ABCDEFGH$ on the coordinates axes that its sides are 2 units in length.



The coordinates of octagon vertexes are clear, because based on the equation $x^2 + x^2 = 4$ the value of x is $\sqrt{2}$. Also, the coordinates of quadrilateral vertexes are determinable easily, since the points W, X, Y, and Z are the midpoints of the sides AB, CD, EF, and GH, respectively. As respect to the coordinates of points $X(2\sqrt{2}+2, 1+\sqrt{2})$, $Y(1+\sqrt{2}, 0)$ and $Z(0, 1+\sqrt{2})$, the slope of sides XY and ZY are calculated as follows:

$$a_{XY} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+\sqrt{2}) - 0}{(2\sqrt{2}+2) - (1+\sqrt{2})} = 1$$
$$a_{ZY} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1+\sqrt{2}) - 0}{0 - (1+\sqrt{2})} = -1$$

It shows $XY \perp ZY$ $(a_{XY}a_{ZY} = 1(-1) = -1)$ therefore, $WXYZ$ is a square. The length of square side is $ZY = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(1+\sqrt{2}-0)^2 + (0-1-\sqrt{2})^2} = \sqrt{4\sqrt{2}+6}$. The surface of this square is $S_{WXYZ} = ZY^2 = (\sqrt{4\sqrt{2}+6})^2 = 4\sqrt{2}+6$. The surface of regular octagon is determined as the following:

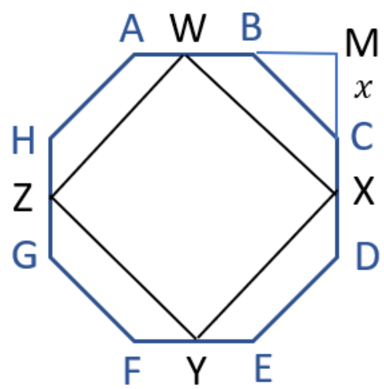
$$S_{ABCDEFGH} = (2+2\sqrt{2})^2 - 4\left(\frac{\sqrt{2}\times\sqrt{2}}{2}\right) = 8 + 8\sqrt{2}.$$

Therefore, the probability that a point chosen uniformly at random from inside the octagon $ABCDEFGH$ will be located inside the quadrilateral $WXYZ$ is calculated using the probability formula as below:

$P = \frac{S_{WXYZ}}{S_{ABCDEFGH}} = \frac{4\sqrt{2}+6}{8+8\sqrt{2}} = \frac{\sqrt{2}+1}{4}.$

Solution 2:
Based on the following figure, the surface of regular octagon is determined as the following:

$S_{ABCDEFGH} = (2+2\sqrt{2})^2 - 4\left(\frac{\sqrt{2}\times\sqrt{2}}{2}\right) = 8+8\sqrt{2}.$



In an isosceles triangle WMX, angle MWX is 45 degrees. In a similar way, it can be shown that the angle AWZ is 45 degrees. As a result, angle ZWX is 90 degrees, which shows that the quadrilateral $WXYZ$ is a square. The length of the side of this square is calculated as follows.

$WX^2 = WM^2 + MX^2 = (1+\sqrt{2})^2 + (1+\sqrt{2})^2 = 6+4\sqrt{2}.$

Therefore, the surface of this square is $S_{WXYZ} = WX^2 = 6+4\sqrt{2}$. The desired probability is obtained as follows.

$P = \frac{S_{WXYZ}}{S_{ABCDEFGH}} = \frac{4\sqrt{2}+6}{8+8\sqrt{2}} = \frac{\sqrt{2}+1}{4}.$

Second Solution to problem 29

By Dr. Abdullah Kurudirek, Mathematics Ed. Department, Tishk International University.

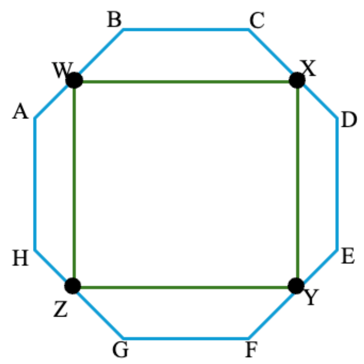
Our solver provides a concise solution, relying on familiar formulas for regular polygons, specifically the octagon and square. By combining side lengths, areas, and symmetry, the ratio of areas is calculated, which leads to determining the desired probability. A graph is included to further clarify the explanation.

Solution 29

As it is known, the area of a regular octagon is $2a^2(1+\sqrt{2})$ where a is the side length.
For $a = 2$, therefore

$$A(ABCDEFGH) = 8(1+\sqrt{2})$$

According to the given information, when the midpoints of alternating sides of a regular octagon are joined, we get a square WXYZ. If one side of the regular octagon is 2, then one side of the square will be $2+\sqrt{2}$.



Let P be the probability that a point chosen uniformly at random from inside the octagon ABCDEFGH will be located inside the quadrilateral WXYZ then

$$P = \frac{Area(WXYZ)}{Area(ABCDEFGH)} = \frac{(2+\sqrt{2})^2}{8(1+\sqrt{2})} = \frac{1+\sqrt{2}}{4}$$

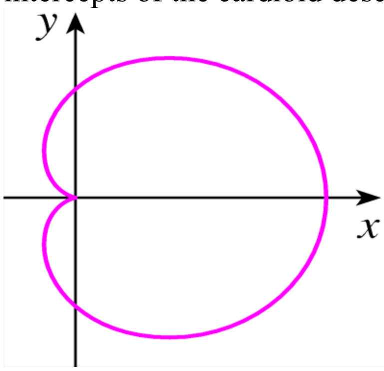
Dear fellow problem solvers,

I'm glad you enjoyed tackling problems 28 and 29 and that you've expanded your approach to mathematics. Let's jump into the next set of problems to keep advancing your skills.

Problem 30

Proposed by Ivan Retamoso, BMCC, USA.

Find the coordinates of the intersection point of the tangent lines at the highest and lowest y-intercepts of the cardioid described by $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$.



Problem 31

Proposed by Ivan Retamoso, BMCC, USA.

In the figure below, $AD = CF = 8\ cm$, $DB = BC = 2\ cm$, and the area of triangle ABC is $7.2\ \ cm^2$. Find the area of the quadrilateral $DECB$.