The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem and its solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I am delighted to announce that I have received solutions to both Problem 14 and Problem 15. I am thrilled to report that all of them were not only correct but also truly fascinating and innovative. By showcasing what I consider to be the most outstanding solutions, my primary goal is to enrich and elevate the mathematical understanding of our global community.

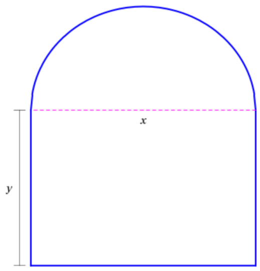
Solutions to **Problems** from the Previous Issue.

Interesting "Optimal configuration" problem.

Problem 14

Proposed by Ivan Retamoso, BMCC, USA.

Let's imagine a scenario where a corral is being enclosed using 130 ft of fencing. The corral is in the shape of a rectangle, and it has a semicircle attached to one of its sides. The diameter of the semicircle aligns with the length of the rectangle, as depicted in the figure provided.

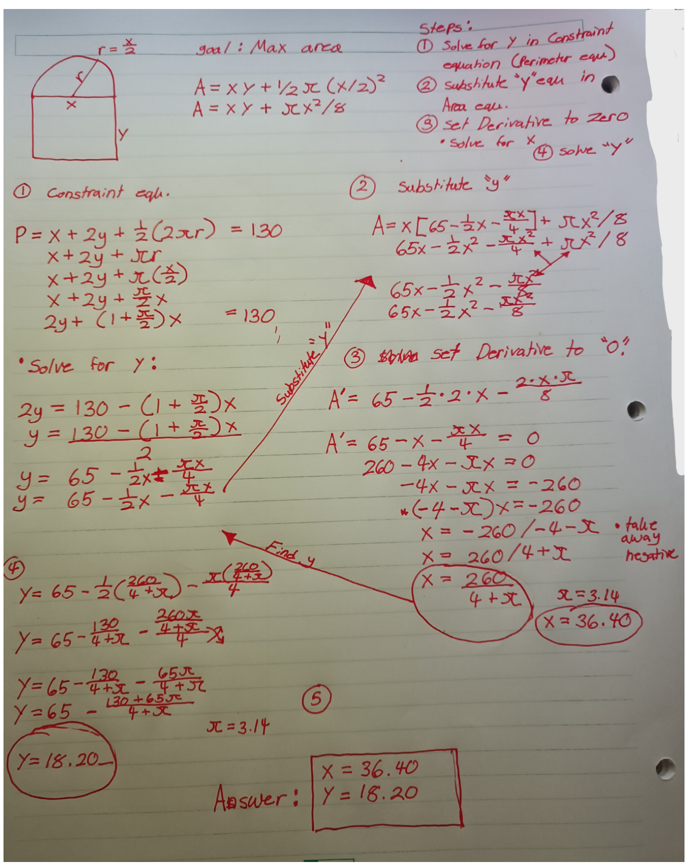


Determine the values of x and y that will result in the corral having the largest possible area.

Solution to problem 14

By Jack Powell, Borough of Manhattan Community College, USA.

In this initial solution, our solver succinctly presents the objective function and expertly transforms it into a single-variable expression using the constraint equation. Moreover, the application of calculus plays a pivotal role as our solver skillfully sets the derivative of the objective function to zero and proceeds to resolve for the variable.

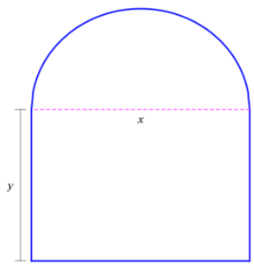


Second Solution to problem 14

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This alternative approach follows a similar strategy, but with a meticulous attention to detail, ensuring no steps are skipped. Our solver presents the solutions in exact form, offering valuable insights into the structure of the final solution. In other words, it reveals the precise proportions of the dimensions x and y that lead to the maximum area of the corral. This level of accuracy enables a better understanding of the corral's optimal configuration.

Solution: Consider the below diagram



As per question we have

Area of the Corral = Area of the rectangle + Area of the Semicircle

$$= \quad xy \quad + \quad \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

Area = $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$ ......(1)

Perimeter of the Corral = $x + 2y + \frac{1}{2}2\pi\left(\frac{x}{2}\right) = 130$

Hence, we have

$x + 2y + \pi\left(\frac{x}{2}\right) = 130$

$y = 65 - \frac{x}{2} - \pi\left(\frac{x}{4}\right) = 65 - x\left(\frac{2+\pi}{4}\right)$ ......(2)

Substituting the value of $y = 65 - x\left(\frac{2+\pi}{4}\right)$ in equation given by (1) we get

Area = $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$= x\left\{65 - x\left(\frac{2+\pi}{4}\right)\right\} + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$

$= 65x - x^2\left(\frac{2+\pi}{4}\right) + \pi\frac{x^2}{8}$

$= -x^2\left(\frac{4+\pi}{8}\right) + 65\, x$

Notice: Area above is a function of x and it represents parabola opening downwards. The vertex of the parabola is the point of maxima. The x-coordinate of the vertex of the parabola is $\frac{-65}{-2\left(\frac{4+\pi}{8}\right)}$

$= \frac{65}{2\left(\frac{4+\pi}{8}\right)} =$

$\frac{65\times4}{4+\pi} = \frac{260}{4+\pi}$

(The general equation of a parabola is given as $Ax^2 + \mathrm{B}\, x + \mathrm{C} = 0$, the x-coordinate of the vertex is given by $\left(\frac{-B}{2A}\right)$).

Substituting the value of x in the equation given by (2) we get

$y = 65 - x\left(\frac{2+\pi}{4}\right)$

$= 65 - \left[\frac{260}{(4+\pi)} \times \left(\frac{2+\pi}{4}\right)\right] = 65 - \left[\frac{65}{(4+\pi)} \times (2+\pi)\right]$

$= \frac{[65\,(4+\pi)]-[65\times(2+\pi)]}{(4+\pi)}$

$= \frac{(65\times4)+(65\times\pi)-(65\times2)-(65\times\pi)}{(4+\pi)}$

$= \frac{(65\times4)-(65\times2)}{(4+\pi)}$

$= \frac{(65\times2)}{(4+\pi)}$

$= \frac{(130)}{(4+\pi)}$

The Corral will have the largest possible area when $x = \frac{260}{(4+\pi)}$ and $y = \frac{130}{(4+\pi)}$

Tricky algebra problem.

Problem 15

Proposed by Ivan Retamoso, BMCC, USA.

x, y, and z are real numbers such that $x+y+z=17$ and $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}=\frac{4}{15}$ find the exact value of $\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$.

Solution to problem 15

By Aradhana Kumari, Borough of Manhattan Community College, USA.

Initially, we encounter a challenging system of equations with two equations and three variables, seemingly impossible to solve directly. However, using an ingenious algebraic trick, our solver managed to compute the value of the desired mathematical expression. I invite you to witness this remarkable solution for yourself.

Solution: Consider the equation given in the problem

$$\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}=\frac{4}{15}$$

Multiply above equation by $(x+y+z)$ we get

$$\left\{\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}=\frac{4}{15}\right\}\times\ (x+y+z)$$

$$\frac{x+y+z}{x+y}+\frac{x+y+z}{y+z}+\frac{x+y+z}{z+x}=\frac{4\,(x+y+z)}{15}$$

After rearranging the terms, we get

$$\frac{x+y}{x+y}+\frac{z}{x+y}+\frac{y+z}{y+z}+\frac{x}{y+z}+\frac{z+x}{z+x}+\frac{y}{z+x}=\frac{4\,(x+y+z)}{15}$$

$$1+\frac{z}{x+y}+1+\frac{x}{y+z}+1+\frac{y}{z+x}=\frac{4\,(x+y+z)}{15}$$

$$3+\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{4\,(x+y+z)}{15}$$

$$\frac{z}{x+y}+\frac{x}{y+z}+\frac{y}{z+x}=\frac{4\,(x+y+z)}{15}-3$$

Substituting the value of $x+y+z=17$ in the above equation and rearranging the terms we get

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{4\times 17}{15}-3$$

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{68}{15}-3=\frac{68-45}{15}=\frac{23}{15}$$

hence

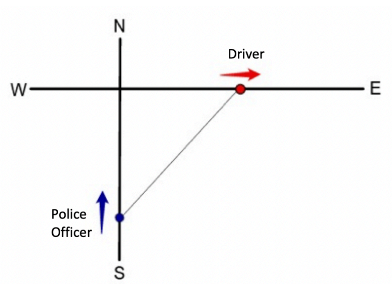
$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=\frac{23}{15}$$

Dear fellow problem solvers,

I trust that tackling problems 14 and 15 not only brought you enjoyment but also provided valuable insights. Now, let's proceed to the next two problems to continue this rewarding journey of exploration and learning.

Problem 16

Proposed by Ivan Retamoso, BMCC, USA.



Let's consider a situation where a police officer is situated $\frac{1}{2}$ mile to the south of an intersection. This officer is driving northwards towards the intersection at a speed of 35 mph. At the exact same time, there is another car located $\frac{1}{2}$ mile to the east of the intersection, and it is moving eastward, away from the intersection.

a) Let's assume that the officer's radar gun displays a speed of $20\ mph$ when aimed at the other car. This reading indicates that the straight-line distance between the officer and the other car is increasing at a rate of $20\ mph$. What, then, is the speed of the other car?
b) Now, let's consider a different scenario where the officer's radar gun displays $-20\ mph$ instead. This indicates that the straight-line distance between the officer and the other car is decreasing at a rate of $20\ mph$. What is the speed of the other car in this situation?

Note: Round yours answers to three decimals places.

Problem 17

Proposed by Christopher Ingrassia, Kingsborough Community College (CUNY)
Brooklyn, NY, USA

Suppose $n \times n$ matrix A and $n \times 1$ vector x are defined as follows:

$$A_{i,j} = \begin{cases} 1, & i \geq j \\ 0, & otherwise \end{cases}$$

$$x_i = 1$$

Describe, in words, the vector $A^k x$, where $k \geq 0$.

Find an expression for the quantity $x^T A^k x$ in terms of n and k (x^T is the transpose of vector x).