The Problem Corner

Ivan Retamoso, PhD, The Problem Corner Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, solutions to **Problem 1** were submitted, and I am glad to report that they were correct and interesting. The solutions followed different approaches, which I hope will enrich and enhance the mathematical knowledge of our community.

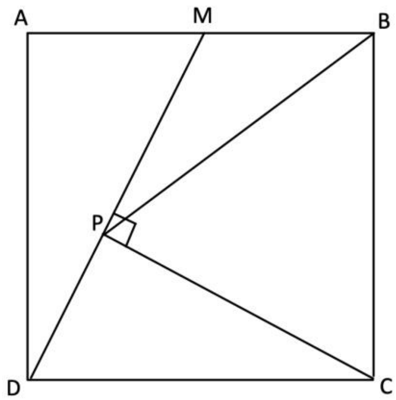
Solutions to **Problem 1** from a Previous Issue

Interesting Square Problem

Proposed by Ivan Retamoso

Problem 1

Let $ABCD$ be a square with side length 1 $meter$ and let M be the midpoint of the segment AB. While connecting D and M, draw a perpendicular segment from point C to the segment DM. Marking as P the point of the intersection of the perpendicular and the segment DM. Find the length of the segment PB.



Solution 1 by Aradhana Kumari, Borough of Manhattan Community College.

This solution is mainly based on Trigonometric properties such as The Law of Cosines, together with Tangent Inverse, it is important to notice that this solution does not require auxiliary lines, which makes it appealing to readers which consider auxiliary lines hard to "figure out", additionally, the "exact form" of the solution could lead to a general formula for the length of the segment PB for an arbitrary side length of a given square.

Solution: From the right triangle MAD,

$\tan\angle ADM = \frac{length\ of\ side\ AM}{lenght\ of\ side\ AD} = \frac{1/2}{1}$

Therefore ∠ADM = $\tan^{-1}(1/2)$

hence ∠PDC = $90° - \tan^{-1}(1/2)$….(1)

Now consider the right triangle DPC,

$\angle\mathrm{PDC} = 90° - \tan^{-1}(1/2)$,

$hence$ ∠PCD = $\tan^{-1}(1/2)$,

$\angle$PCB = $90° \; - \tan^{-1}(1/2)$.... (2)

Again, consider the right triangle DPC

We have Sin $\angle \mathrm{PDC} \;=\; \mathrm{Sin}\,(90° - \tan^{-1}(1/2))$ = $\frac{length\ of\ side\ PC}{lenght\ of\ side\ DC}$ = $\frac{length\ of\ side\ PC}{1}$,

Therefore, the length of side PC = Sin $(90° - \tan^{-1}(1/2))$......(3)

Now consider the triangle PCB, using the cosine rule,

(length of side PB)^2

= (length of side BC)^2+ (length of side CP)^2 - 2 (length of side BC)×(length of side CP) Cosine ∠BCP,

therefore, Length of side PB =

$$\sqrt{(\text{length of side BC})^2 + (\text{length of side PC})^2 - 2\,(\text{length of side BC}) \times (\text{ length of side PC}) \times \text{Cosine } \angle\text{BCP}}$$

.... (4)

Substituting the value of length of side BC (= 1),

length of side PC = Sin ($90°$- $\tan^{-1}(1/2)$) , (from equation (3)) ,

and measure of ∠BCP = $90° \; - \tan^{-1}(1/2) \;\; (from\ equation\ 2)$,

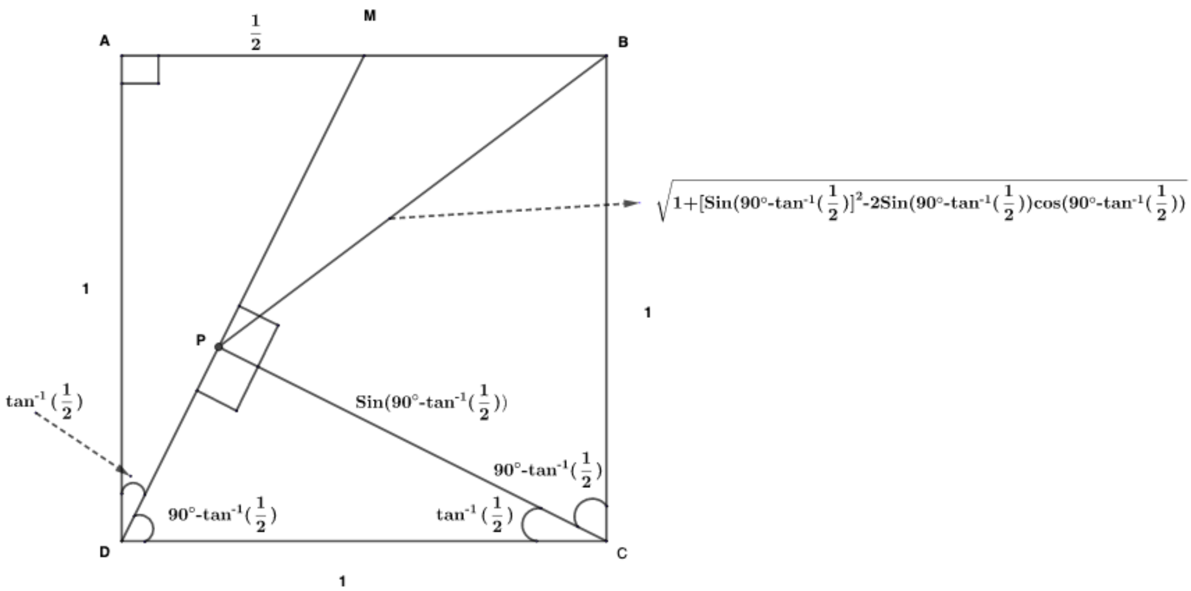
in equation given by (4) we get length of side PB

= length of segment PB

$$= \sqrt{1 + [\text{Sin}\,(90° - \tan^{-1}(1/2))]^2 - 2 \times 1 \times \text{Sin}\,(90° - \tan^{-1}(1/2)) \times \text{Cos}(90° - \tan^{-1}(1/2))}$$

$$= \sqrt{1 + [\text{Sin}\,(90° - \tan^{-1}(1/2))]^2 - 2\,\text{Sin}\,(90° - \tan^{-1}(1/2)) \times \text{Cos}(90° - \tan^{-1}(1/2))}$$

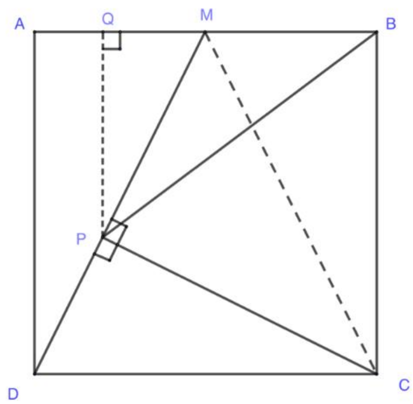
$$= 1$$



Solution 2 by Aradhana Kumari, Borough of Manhattan Community College.

This second solution uses Geometric properties such as the theorem of Pythagoras and two auxiliary lines together with Heron's formula for the computation of the area of a triangle, this solution serves to show that different paths can takes us to the same truth.

Solution:



Consider the right triangle DAM, we have AM = $\frac{1}{2}$ and DA= 1

Since $(AM)^2 + (DA)^2 = (DM)^2$

Hence $(DM)^2 = (\frac{1}{2})^2 + (1)^2 = \frac{1}{4} + 1 = \frac{5}{4}$

Therefor DM = $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$ .....(1)

Consider the right triangle CBM, we have

CM = $\sqrt{(1)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$Consider the triangle DMC, DM = $\frac{\sqrt{5}}{2}$, CM = $\frac{\sqrt{5}}{2}$, DC = 1

Since Area of the triangle DMC

= $\frac{DM \times PC}{2} = \sqrt{S\,(S-a)\,(S-b)\,(S-c)}$ ..... (2)

where S = $\frac{(a+b+c)}{2}$,

a = length of side DM

b = length of side CM

c = length of side DC

S = $\frac{[\frac{\sqrt{5}}{2}+\frac{\sqrt{5}}{2}+1]}{2}$ = $\frac{1+\sqrt{5}}{2}$

Substituting the value of S, a, b, and c in equation given by (2) we get

$\frac{DM \times PC}{2}$ =

$$=\sqrt{(\frac{1+\sqrt{5}}{2})\,(\frac{1+\sqrt{5}}{2}-\frac{\sqrt{5}}{2})\,(\frac{1+\sqrt{5}}{2}-\frac{\sqrt{5}}{2})\,(\frac{1+\sqrt{5}}{2}-1)}$$

$$=\sqrt{(\frac{1+\sqrt{5}}{2})\,(\frac{1}{2})\,(\frac{1}{2})\,(\frac{\sqrt{5}-1}{2})}$$

$$=\sqrt{(\frac{1}{16})(\sqrt{5}-1)\,(\sqrt{5}+1)}$$

$$=\frac{1}{4}\sqrt{(\sqrt{5}-1)\,(\sqrt{5}+1)}$$

$$=\frac{1}{4}\sqrt{(\sqrt{5})^2-(1)^2}=\frac{1}{4}\sqrt{4}=\frac{2}{4}=\frac{1}{2}$$

$$\frac{DM \times PC}{2} = \frac{1}{2}$$

$$\text{DM} \times PC = 1$$

$$\text{PC} = \frac{1}{DM} = \frac{1}{\frac{\sqrt{5}}{2}} = \frac{2}{\sqrt{5}}$$

Consider the right triangle DPC, we have $(DP)^2 + (PC)^2 = (DC)^2$

Therefore DP = $\sqrt{(1)^2 - \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{1 - \frac{4}{5}} = \frac{1}{\sqrt{5}}$

Since DM = DP+ PM

PM = DM – DP

$$= \frac{\sqrt{5}}{2} - \frac{1}{\sqrt{5}} = \frac{5-2}{2\sqrt{5}} = \frac{3}{2\sqrt{5}}$$

PM = $\frac{3}{2\sqrt{5}}$ ……. (3)

Draw a perpendicular from P on the segment AM, where it intersects the segment AM call it Q.

Consider the right triangle DAM and right triangle PQM, since they both have the same acute angle, angle DAM (or angle PMQ) hence right triangle DAM and right triangle PMQ are similar triangle, Therefore $\frac{DA}{PQ} = \frac{DM}{PM}$

Hence PQ = $\frac{DA \times PM}{DM}$

substituting the value DA = 1, PM = $\frac{3}{2\sqrt{5}}$ (from (3)), DM = $\frac{\sqrt{5}}{2}$ (from (1)) we have

Hence PQ = $\frac{DA \times PM}{DM}$ = $\frac{1 \times \frac{3}{2\sqrt{5}}}{\frac{\sqrt{5}}{2}}$ = $\frac{3}{2\sqrt{5}} \times \frac{2}{\sqrt{5}}$ = $\frac{3}{5}$

PQ = $\frac{3}{5}$ …… (4)

Consider again the similar triangle DAM and PQM

We have $\frac{DA}{PQ} = \frac{AM}{QM}$

Therefore QM = $\frac{AM \times PQ}{DA}$, substituting the value of AM = $\frac{1}{2}$, PQ = $\frac{3}{5}$ (from (4)), DA= 1, we have

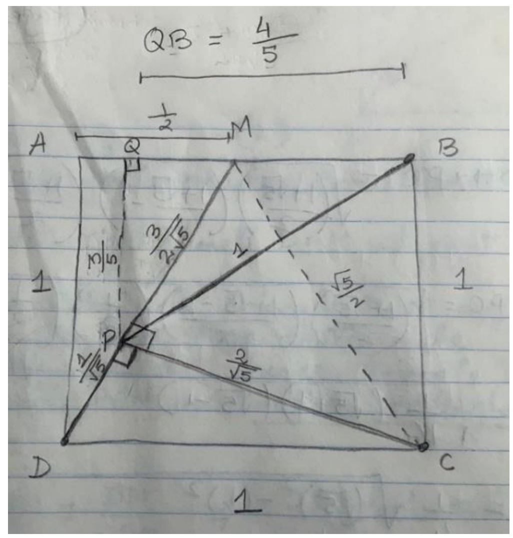
QM = $\frac{\frac{1}{2} \times \frac{3}{5}}{1}$ = $\frac{3}{10}$ …..(5)

Since QB = QM + MB, since QM = $\frac{3}{10}$, and MB = $\frac{1}{2}$ (since M is the midpoint of the side AB hence MB = $\frac{1}{2}$)

$= \frac{3}{10} + \frac{1}{2} = \frac{8}{10} = \frac{4}{5}$

QB = $\frac{4}{5}$ ……. (6)

Consider the right triangle PQB, we have



$(PQ)^2 + (QB)^2 = (PB)^2$

Therefore PB = $\sqrt{(PQ)^2 + (QB)^2}$

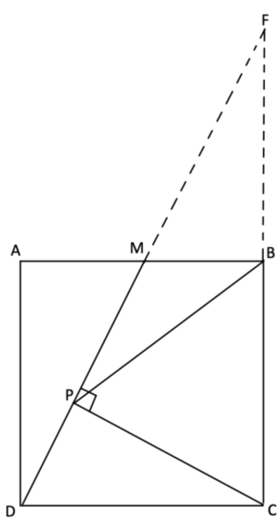
$= \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})} = \sqrt{\frac{9}{25} + \frac{16}{25}}$

$$= \sqrt{\frac{25}{25}} = 1$$

Therefore, the length of segment PB = 1 meter

Solution 3 by the proposer Ivan Retamoso, Borough of Manhattan Community College.

This solution uses basic facts from Geometry and 2 auxiliary lines, even though it is difficult to figure out the auxiliary lines needed to solve the problem, it shows that, in some cases, basic properties could lead to remarkable results.



Extend DM and CB until they intersect at point F. MB is parallel to DC, the length of MB is half the length of DC, $\triangle MBF$ is similar to $\triangle DCF$ then B is midpoint of CF and PB is the median of the right triangle CPF, since in a right triangle the length of the median to the hypothenuse is equal to half the length of the hypothenuse then the length of PB is $1\ meter$.

Also solved by Jesse Wolf.

I hope you enjoyed solving Problem 1, below is the next problem.

Problem 2

The distance between Paul's home and his School is 2 miles. One day after his school day is over, Paul decides to walk back home by taking 2 straight paths perpendicular to each other, assume the territory where Paul's home and his school are located allows him to do it, any way he wishes as long as the 2 straight paths are perpendicular to each other.

a) What is the total length of the largest path Paul can take to go back home from his school?
b) Give a compass and straightedge construction of the path you found in part a) starting from the distance between Paul's home and his School which is 2 miles, using any scale to represent a mile.