The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, I got solutions to **Problem 3**, and I am happy to inform that they were correct, interesting, and ingenious. By posting different solutions, I hope to enrich and enhance the mathematical knowledge of our international community.

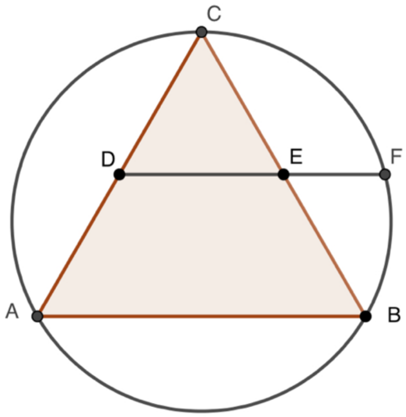
Solutions to **Problem** from the Previous Issue

Interesting Geometric Problem with a surprising solution.

Proposed by Aradhana Kumari Borough of Manhattan Community College, City university of New York, USA

Problem 3

Triangle ABC is an equilateral triangle inscribed in a circle. D and E are the mid points of sides AC and BC respectively. Find the ratio, length DF : length DE?

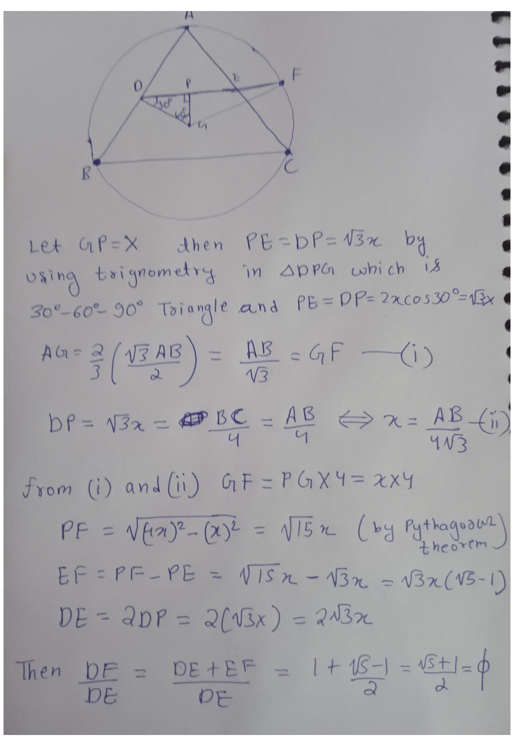


Solution 1

by Jayendra Jha, Arihant Public School, India

and Sankalp Savaran, Shiv jyoti Senior Secondary School, India.

This solution, interestingly, combines Geometry (Centroid property and the Theorem of Pythagoras) and Trigonometry (sine and cosines of angles 30°, and 60°), and some basic algebra, ending up with the solution in exact form.



Let GP = X then PE = DP = $\sqrt{3}x$ by using trignometry in ΔDPG which is 30°-60°-90° Triangle and PE = DP = $2x\cos 30° = \sqrt{3}x$

$$AG = \frac{2}{3}\left(\frac{\sqrt{3}AB}{2}\right) = \frac{AB}{\sqrt{3}} = GF \quad \text{—(i)}$$

$$DP = \sqrt{3}x = \frac{BC}{4} = \frac{AB}{4} \iff x = \frac{AB}{4\sqrt{3}} \quad \text{—(ii)}$$

from (i) and (ii) $GF = PG \times 4 = x \times 4$

$$PF = \sqrt{(4x)^2 - (x)^2} = \sqrt{15}x \quad \text{(by Pythagoras theorem)}$$

$$EF = PF - PE = \sqrt{15}x - \sqrt{3}x = \sqrt{3}x(\sqrt{5}-1)$$

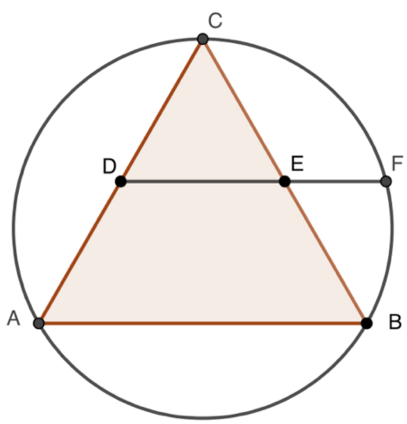
$$DE = 2DP = 2(\sqrt{3}x) = 2\sqrt{3}x$$

Then $\frac{DF}{DE} = \frac{DE+EF}{DE} = 1 + \frac{\sqrt{5}-1}{2} = \frac{\sqrt{5}+1}{2} = \phi$

Solution 2

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA (The proposer).

This solution is based on a clever change of variable, an auxiliary extension of a segment together with the Intersecting chords theorem.



Let the length of sides of the equilateral triangle ABC as 2x. Since D is the midpoint of CA and E is the midpoint of CB therefore the length of CD and CE is x.

In the triangle CDE,

length CD = length CE = x

hence *angle* CDE = angle CED

since the angle DCE is 60°

angle CDE + angle CED + 60° = 180°

angle CDE + angle CDE + 60° = 180°

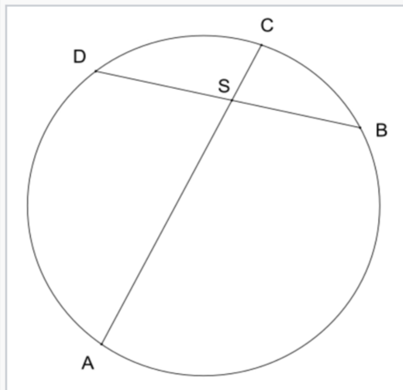
2 × angle CDE = 120°

Angle CDE = 60°

Hence triangle CDE is an equilateral triangle with side lengths x.

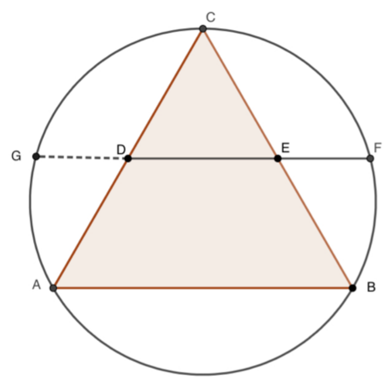
Let the length of EF = y then length of DF = x+y and length of DG = y.

Intersecting chord theorem: If two chords intersect in a circle, then the products of the measures of the segments of the chords are equal. (The below picture is taken from Wikipedia.)



$|AS| \cdot |SC| = |BS| \cdot |SD|$

In the below diagram the two chords GF and CA are intersecting at D.



Hence by intersecting chord theorem, we have

length of GD × length DF = length CD × length DA

y (x+y) = x · x

$\frac{(x+y)}{x} = \frac{x}{y}$

$\frac{x}{x} + \frac{y}{x} = \frac{x}{y}$ …….. (1)

Substitute $\frac{x}{y}$ = α in the above equation given by (1)

We get 1+ $\frac{1}{\alpha}$ = α

After simplifying we get α + 1 = α^2

or α^2 - α – 1 = 0

therefore $\alpha = \frac{1+\sqrt{5}}{2}$ hence $\frac{x}{y} = \frac{1+\sqrt{5}}{2}$

Therefore

$\frac{\boldsymbol{lenght\ DF}}{\boldsymbol{lenght\ DE}} = \frac{\boldsymbol{x+y}}{\boldsymbol{x}} = \frac{\boldsymbol{x}}{\boldsymbol{y}} = \frac{\mathbf{1+\sqrt{5}}}{\mathbf{2}}$

$Note$: $\frac{1+\sqrt{5}}{2}$ is also known as golden ratio.

Solution 3

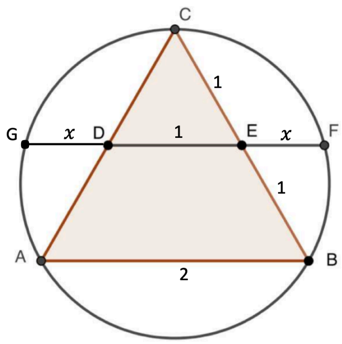
by Ivan Retamoso, Borough of Manhattan Community College, USA (Editor of *The Problem Corner*).

This solution uses an auxiliary line and exploits the symmetry and the independence of the units of measurarements, since the solution is a ratio.

Since we are looking for a ratio, without loss of generality, let the side length of the equilateral tringle be 2 units.

Then $\overline{AB} = 2$, $\overline{DE} = 1$, $\overline{EC} = 1$, and $\overline{EB} = 1$

Let's extend DE to the left, where DE meets the circle let's call this point G, let x be the length EF and GD which are the same due to Symmetry as shown in the figure below



By The Intersecting Chords Theorem

$\overline{GE} \cdot \overline{EF} = \overline{CE} \cdot \overline{EB}$

$(x+1) \cdot x = 1 \cdot 1$

$x^2 + x = 1$

$x^2 + x - 1 = 0$

$$x = \frac{-1+\sqrt{5}}{2}$$

Then

$$\frac{\overline{DF}}{\overline{DE}} = \frac{1 + \frac{-1+\sqrt{5}}{2}}{1}$$

Then

$$\frac{\overline{DF}}{\overline{DE}} = \frac{1+\sqrt{5}}{2}$$

Note:

The number $\frac{1+\sqrt{5}}{2}$ is "The Golden Ratio", amazing!

Dear Problem Solvers,

I really hope you enjoyed solving Problem 3 as much as I did, below are the next two problems, I am happy to tell you that a Canadian professor has proposed a "proof" problem, I must warn you it is a little advance, but it is accompanied with hints and graphs as help.

Problem 4

Proposed by Ivan Retamoso, BMCC, USA

In a cartesian plane, between the half of the parabola $y = \frac{x^2}{2}$ for $x \geq 0$ and the x – axis there is a circle tangent to the parabola at the point $(2,2)$ and to the x – axis, find the radius of the circle.

Problem 5

Proposed by Mohsen Soltanifar, Adjunct Instructor, Continuing Studies Division, University of Victoria, Victoria, BC, Canada

Let $f(x) = x^x$ $(x > 0)$ be the second tetration function. Prove that f is continuous merely using the $\epsilon - \delta$ definition.

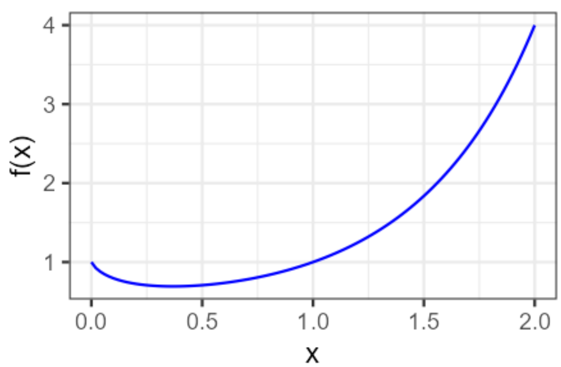


Figure 1: The plot of the second tetration function $f(x) = x^x (x > 0)$.

Hint:

Step (i) Prove that the logarithm function $ln(.)$ is continuous using the $\epsilon - \delta$ definition, and save $\delta = \delta(\epsilon)$.

Step (ii) Prove that the exponential function $exp(.)$ is continuous using the $\epsilon - \delta$ definition, and save $\delta = \delta(\epsilon)$.

Step (iii) Prove that if the function $g(.)$ Is continuous at $x = a$ and the function $f(.)$ Is continuous at $y = g(a)$, then the function $f \circ g(.)$ Is continuous at $x = a$, using the $\epsilon - \delta$ definition.

Step (iv) Use steps (i),(ii) and (iii) for $f(x) = exp(x)$, and $g(x) = x\, ln(x)$ in reversed method to prove the statement for the second tetration function.

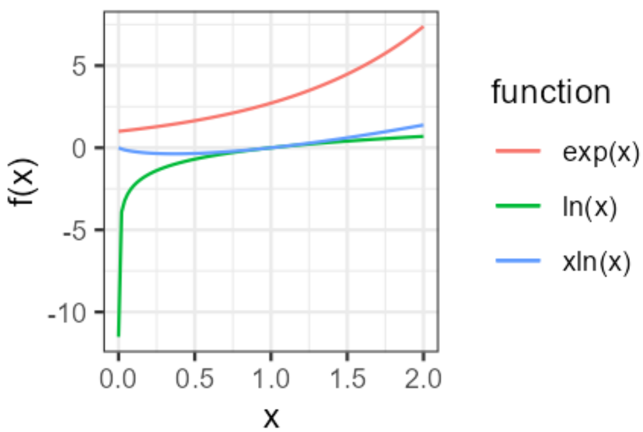


Figure 2: The plot of the three functions $f(x) = exp(x), ln(x), x\, ln(x), (x > 0)$.