The Problem Corner

Ivan Retamoso, PhD, The Problem Corner Editor

Borough of Manhattan Community College

iretamoso@bmcc.cuny.edu

The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, solutions to **Problem 2** were submitted, and I am glad to report that they were correct and interesting. The solutions followed different approaches, which I hope will enrich and enhance the mathematical knowledge of our community.

Solutions to **Problem** from a Previous Issue

Interesting "Largest path" Problem

Proposed by Ivan Retamoso

Problem 2

The distance between Paul's home and his School is 2 miles. One day after his school day is over, Paul decides to walk back home by taking 2 straight paths perpendicular to each other, assume the territory where Paul's home and his school are located allows him to do it, any way he wishes as long as the 2 straight paths are perpendicular to each other.

a) What is the total length of the largest path Paul can take to go back home from his school?
b) Give a compass and straightedge construction of the path you found in part a) starting from the distance between Paul's home and his School which is 2 miles, using any scale to represent a mile.

Solution 1

by Aradhana Kumari, Borough of Manhattan Community College, USA.

This solution uses Calculus via Differentiation, the length of the path was found in general and by setting its derivative equal to zero the maximization of the length of the path was accomplished, this is followed by a step-by-step construction of the optimal path using only straight edge and compass.

Solution:

Part a)

As per question we have

Total length L = length SA + length AH

$= 2\cos\theta + 2\sin\theta \;, 0 \leq \theta \leq 90°$

$L' = -2\sin\theta + 2\cos\theta$

To find which angle will maximize the total length consider we have to

Consider the equation $L' = 0$

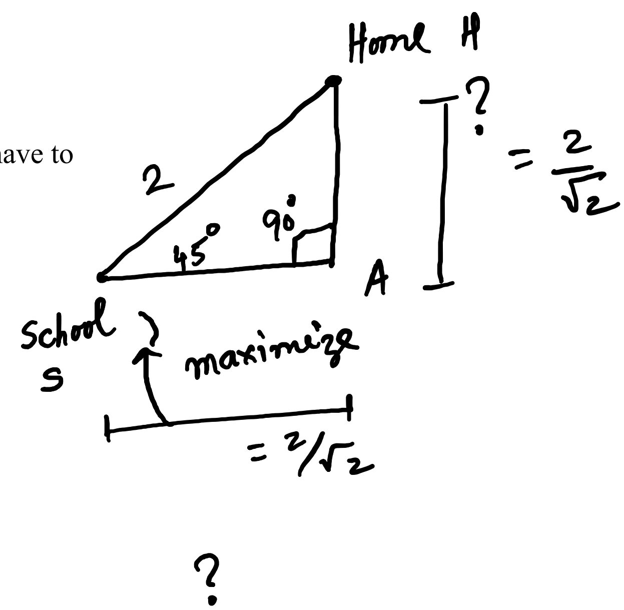
We have $-2\sin\theta + 2\cos\theta = 0$

$-2\sin\theta = -2\cos\theta$

$\tan\theta = 1$

$\theta = 45°$

Therefore, the total length of the largest path is



$2\cos 45° + 2\sin 45° = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

Part b)

Step 1) Draw a line segment of length 4 unit. Call the end points as S (School) and P

Step 2) We will draw the perpendicular bisector of segment SP. Keep the compass at S and draw an arc above and an arc below as shown below.

Step 3) Next move the compass at P and draw an arc above and below as shown below.

Step 4) Join K and L as shown. Segment KL intersect segment SP at the point H at 90°.

Step 5) We will draw the angle bisector of right angle KHS. We keep the compass at H (Home) and draw an arc on the segment HS. Let say this arc intersect the segment SH at point F

Step 6) We keep the compass H and draw an arc on the segment HK. the arc interests the segment HK at point E.

Step 7) We keep the compass at E and draw the arc as shown.

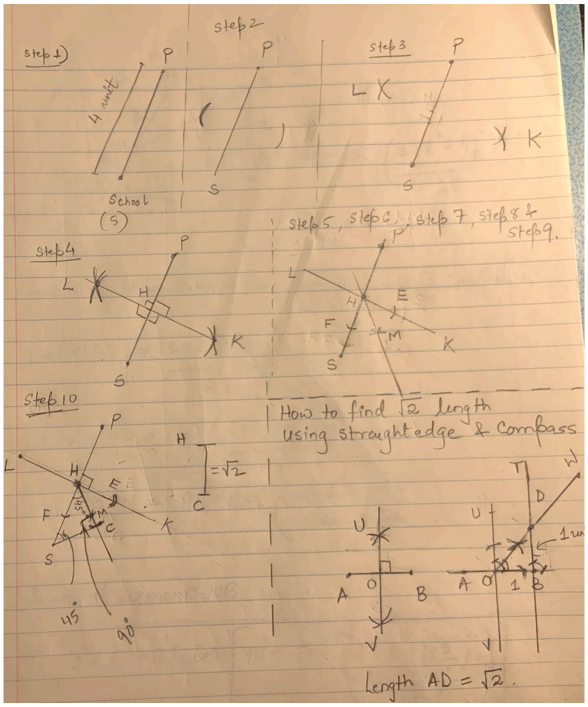
Step 8) We keep the compass at F and draw an arc as shown. These two are met at point M.

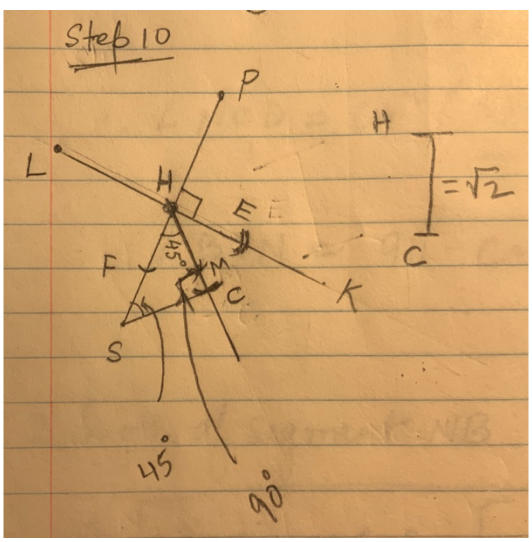
Step 9) We join the point M with H. Angle MHS is 45°.

How to find length $\sqrt{2}$ using compass and straightedge

Consider a line segment AB of length 2 and follow the step 1 through 4 as above to find the perpendicular bisector of this line segment AB. Next, we follow steps like 5,6,7 8 and 9 to bisect angle UOB. We draw a line parallel to line UV passing though B. Let line BT and OW intersect at point D. By construction length BD is 1 unit. Consider the right triangle OBD the length of segment OD is $\sqrt{2}$.

Step 10) Using compass measure segment OD. We put the compass at H and draw an arc of length OD. This arc intersects the line HM at C. Hence, we have length HC is $\sqrt{2}$. We draw a perpendicular line passing through the point S and line HM. Length SC is $\sqrt{2}$.





Solution 2

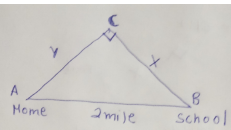
by Jayendra Jha, Arihant Public School, India

and Sankalp Savaran, Shiv jyoti Senior Secondary School, India.

This solution, interestingly, does not use Calculus, instead the Solvers converted the objective function into a form that can be easily maximized using facts from Trigonometry, amazing!

Part a)

Solution :→

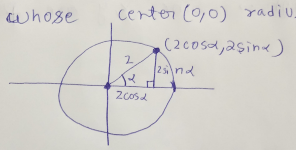


Let A, B be location of Home, school which is 2mile apart.

Paul have to trave distance BC and AC which are perpendicular to each other.

So $x^2+y^2=4$. ——— ①

Here equation ① is equation of circle whose center $(0,0)$ radius is 2.



and question is asking for maximum value of

$$\begin{aligned} x+y &= 2\cos\alpha + 2\sin\alpha \\ &= 2(\cos\alpha + \sin\alpha) \\ &= 2\sqrt{2}\left(\frac{\cos\alpha}{\sqrt{2}} + \frac{\sin\alpha}{\sqrt{2}}\right) \\ &= 2\sqrt{2}\left(\sin\frac{\pi}{4}\cos\alpha + \cos\frac{\pi}{4}\sin\alpha\right) \\ &= 2\sqrt{2}\sin\left(\frac{\pi}{4}+\alpha\right) \end{aligned}$$

Max^m value of $\sin\theta$ when $\theta = \frac{\pi}{4}+\alpha$ is $+1$

so max^m value is $= 2\sqrt{2}$

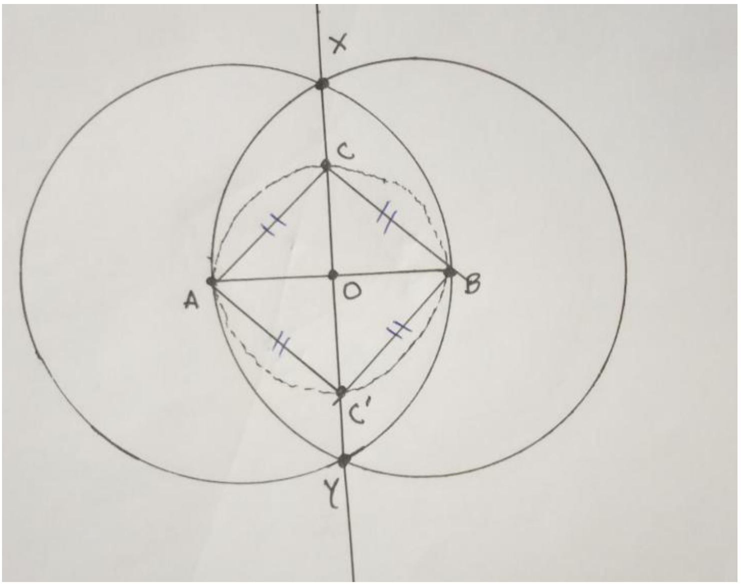
Part b)

In above part (a), the value of α at which $\sin\left(\frac{\pi}{4}+\alpha\right)$ get maximum value is $\frac{\pi}{4}$.

so value of $X = 2\cos\alpha = 2\cos\frac{\pi}{4} = \sqrt{2}$ and

simillarly $Y = 2\sin\alpha = 2\sin\frac{\pi}{4} = \sqrt{2}$ which makes Right isoceles triangle.

Here is the steps and figure to done contruction by straight edge and compass.



<u>Step (i)</u> Let AB = 2miles. Take a compass and Make A as Center and B as Radius and Draw circle T_1.

<u>Step(ii)</u> Make another circle T_2 by considering B as center and A as radius.

<u>Step(iii)</u> Circle T_1 intersect circle T_2 at Two point X & Y.

<u>Step(iv)</u> pass a line through XY which intersect AB at O by using straightedge.

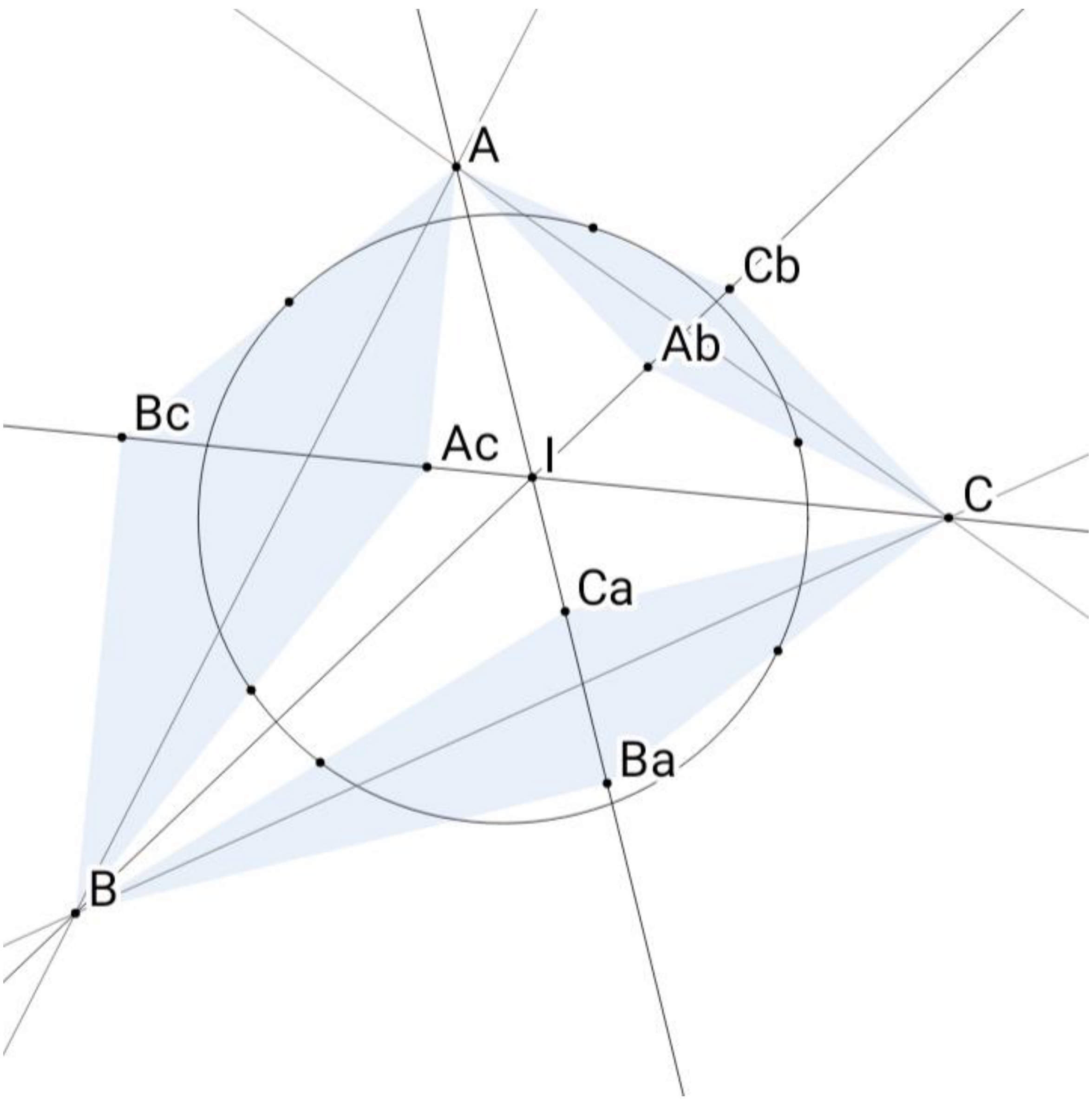
<u>Step(v)</u> make a circle T_3 whose center is O and radius is OA.

<u>Step(vi)</u> Circle T_3 cut ~~the circle~~ Line XY at two point C' and C'.

<u>Hence</u> :→ Path ACB and AC'B becomes two possible ways.

In addition to having solved problem 2, **Jayendra Jha, Arihant Public School, India** and **Sankalp Savaran, Shiv jyoti Senior Secondary School, India** sent me a conjecture they discovered which I wanted to share with all of you for the sake of Mathematical Research.

CONJECTURE: **Let I be incenter of ΔABC and Let Ab,Ac be orthogonal projection from A on Line BI and CI and similarly define {Ba,Bc,Ca,Cb} cyclically then Circumcenter of** (ΔAAcBc);(ΔAAbCb);(ΔBAcBc);(ΔBCaBa);(ΔCBaCa);(ΔCCbAb) **Lies on Same Circle as shown in Given Figure.**



Reference:

[1]Van LamoenCircle https://mathworld.wolfram.com/vanLamoenCircle.html

[2]Discussion of above conjectured with geometers on Euclid messenger group: https://groups.io/g/euclid/topic/88234292#3766

AuthorName:

Sankalp Savaran: savarnsankalp@gmail.com

Jayendra Jha:

archanajha112018@gmail.com

Dear Problem Solvers,

I really hope you enjoyed solving Problem 2, below is the next problem, proposed by Aradhana Kumari, Borough of Manhattan Community College, USA.

Problem 3

Triangle ABC is an equilateral triangle inscribed in a circle. D and E are the mid points of sides AC and BC respectively. Find the ratio, length DF : length DE?