The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, I got solutions to **Problem 4**, and I am happy to inform that they were correct, interesting, and ingenious. By posting what I considered to be the best solutions, I hope to enrich and enhance the mathematical knowledge of our international community.

Solutions to **Problems** from the Previous Issue

Interesting parabola, circle, and common tangent line problem.

Proposed by Ivan Retamoso from Borough of Manhattan Community College, City university of New York, USA

Problem 4

In a cartesian plane, between the half of the parabola $y = \frac{x^2}{2}$ for $x \geq 0$ and the $x-$axis there is a circle tangent to the parabola at the point (2,2) and to the $x-$axis, find the radius of the circle.

Solution to problem 4

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

This solution combines Cartesian Geometry, together with the Derivative from Calculus to find the equation of the tangent line shared by the parabola and the circle, additional algebraic manipulations are cleverly applied as needed, the details of the solution including two explanatory graphs are shown below.

Let the center and radius of the circle described in the above problem be (a,b) and R respectively.

Then the equation of the circle is

$(x-a)^2 + (y-b)^2 = R^2$ ....(1).

Since the above circle is tangent to the x-axis in the first quadrant. Let B be the point where the above circle is tangent on the x-axis. As shown in the Figure 1. the coordinate of the point B is (a,0).

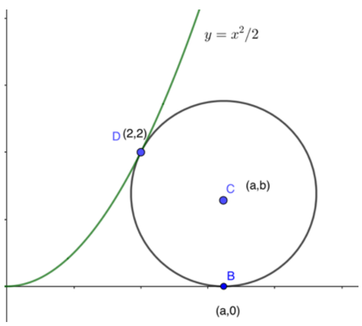

Figure 1.

The slope of the line passing through the center (a,b) of the circle and point (2,2) is $\frac{2-b}{2-a}$

Therefore the slope of the line perpendicular to the line passing through the center (a,b) and point (2,2) is $\frac{-(2-a)}{(2-b)}$.

Note the above

Slope = $\frac{-(2-a)}{(2-b)} = \frac{(a-2)}{(2-b)}$ .... (2)

is the slope of the line tangent to the parabola $y = \frac{x^2}{2}$ (and tangent line to the circle) at the point (2,2).

The equation of the tangent line to the circle at (2,2) is

$(y - 2) = \frac{a-2}{2-b}(x-2)$

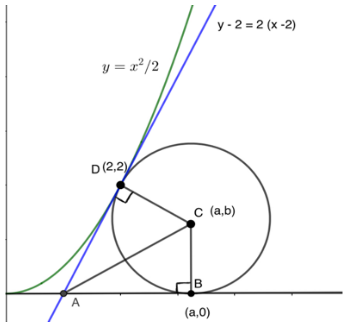

Figure 2.

Consider the given parabola $y = \frac{x^2}{2}$, we have

$\frac{dy}{dx} = \frac{x^2}{2}$ [$\frac{dy}{dx}$ is the slope of the tangent line to the function $y = \frac{x^2}{2}$]

Hence the slope of the tangent line to the parabola $y = \frac{x^2}{2}$, at the point (2,2) is

$\frac{dy}{dx} = \frac{x^2}{2} = \frac{2^2}{2}$ (we put the value of x = 2 in the equation $\frac{dy}{dx} = \frac{x^2}{2}$)

$= \frac{4}{2} = 2$ ....(3)

Hence by equating the value of the slope given by equation (2) and (3) we have

$\frac{(a-2)}{(2-b)} = 2$

or $(a - 2) = 2\,(2 - b)$

therefore $b = \frac{6-a}{2}$ ....(4)

The line (y -2) = 2 $(x - 2)$ is the tangent line to the parabola $y = \frac{x^2}{2}$ and it will intersect the x-axis at the point A $(x, 0)$.

$(0 - 2) = 2\,(x - 2)$ [we substitute $y = 0$]

$-\,2 = 2\,(x - 2)$

Therefore $x = 1$ ....(5)

Hence the coordinate of A is (1,0)

From Figure 2, we have the right triangle ADC and ABC.

In the right triangle ADC, we have

$(AD)^2 + (DC)^2 = (AC)^2$

$(AD)^2 + R^2 = (AC)^2$ (since DC is the radius R)

$(AD)^2 = (AC)^2 - R^2$ ....(6)

In the right triangle ABC, we have

$(AB)^2 + (BC)^2 = (AC)^2$

$(AB)^2 + R^2 = (AC)^2$ (since BC is the radius R)

$(AB)^2 = (AC)^2 - R^2$ ....(7)

Hence from equation given by (6) and (7) we have

$(AD)^2 = (AB)^2$ ....(8)

$(AB)^2 = (a-1)^2$ [The coordinate of A is (1,0) and the coordinate of B is (a,0)]

$(AD)^2 = (2-1)^2 + (2-0)^2 = 5$ [The coordinate of A is (1,0) and the coordinate of D is (2,2)]

Substituting the value of $(AB)^2$ and $(AD)^2$ into the equation given by (8) we have

$(a-1)^2 = 5$

$(a-1) = \pm\sqrt{5}$

$a = 1\pm\sqrt{5}$

As per question coordinate of the point B is positive hence $a = 1+\sqrt{5}$.

From equation given by (4) we have

$$b = \frac{6-a}{2} = \frac{6-(1+\sqrt{5})}{2} = \frac{6-1-\sqrt{5}}{2} = \frac{5-\sqrt{5}}{2}$$

From the Figure 2.

Radius of the circle = distance between the point B and C

$$= \sqrt{(a-a)^2 + (b-0)^2}$$

$$= b$$

$$= \frac{5-\sqrt{5}}{2}$$

Hence the radius of the circle is $\frac{5-\sqrt{5}}{2}$.

Proof problem.

Proposed by Mohsen Soltanifar, Adjunct Instructor, Continuing Studies Division, University of Victoria, Victoria, BC, Canada

Problem 5

Let $f(x) = x^x$ $(x > 0)$ be the second tetration function. Prove that f is continuous merely using the $\epsilon - \delta$ definition.

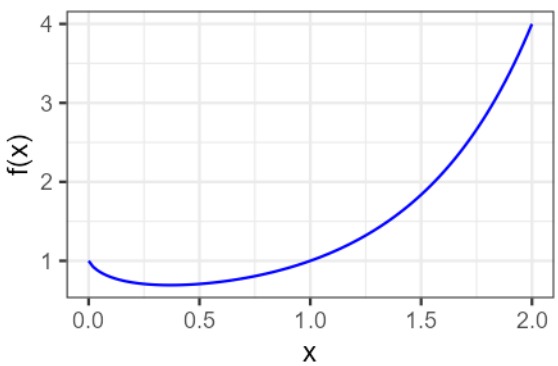

Figure 1: The plot of the second tetration function $f(x) = x^x (x > 0)$.

Hint:

Step (i) Prove that the logarithm function $ln(.)$ is continuous using the $\epsilon - \delta$ definition, and save $\delta = \delta(\epsilon)$.

Step (ii) Prove that the exponential function $exp(.)$ is continuous using the $\epsilon - \delta$ definition, and save $\delta = \delta(\epsilon)$.

Step (iii) Prove that if the function $g(.)$ Is continuous at $x = a$ and the function $f(.)$ Is continuous at $y = g(a)$, then the function $f \circ g(.)$ Is continuous at $x = a$, using the $\epsilon - \delta$ definition.

Step (iv) Use steps (i),(ii) and (iii) for $f(x) = exp(x)$, and $g(x) = x\, ln(x)$ in reversed method to prove the statement for the second tetration function.

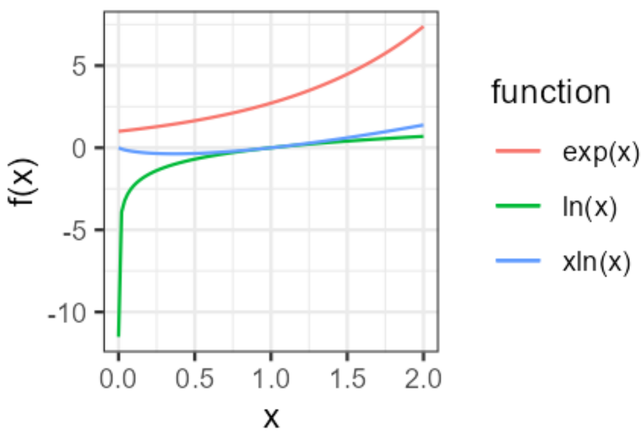

Figure 2: The plot of the three functions $f(x) = exp(x)\,, ln(x)\,, x\,ln(x)\,, (x > 0)$.

Solution to problem 5

By Mohsen Soltanifar, Adjunct Instructor, Continuing Studies Division, University of Victoria, Victoria, BC, Canada (The proposer)

The proof that f is continuous merely using the $\epsilon - \delta$ definition is formally done, step by step starting from showing that the tetration function $f(x) = x^x (x > 0)$ can be seen as the composition of two continuous functions and proving formally that the composition of two continuous functions is continuous, the details of the proof are shown below.

Step(i): A straightforward verification shows that:

$$\lim_{x\to a} ln(x) = ln(a) \colon \forall\, \varepsilon * > 0\colon\ \delta * = a(1 - e^{-\varepsilon *})\ \forall x \in \mathbb{R}\, (|x - a| < \delta * \to |ln(x) - ln(a)| < \varepsilon *). \tag{1}$$

Step(ii): A straightforward verification shows that:

$$\lim_{y\to b} e^y = e^b\ \colon\ \forall\, \varepsilon ** > 0\colon\ \delta ** = ln(1 + \varepsilon **\, e^{-b})\ \forall y \in \mathbb{R}\, (|y - b| < \delta ** \to |e^y - e^b| < \varepsilon **). \tag{2}$$

Step (iii): We want for assigned $a > 0$ and given $\varepsilon > 0$, to find a $\delta > 0$ such that for all $x \in \mathbb{R}$ if $|x - a| < \delta$ then $|f \circ g(x)\ - f \circ g(a)\,| < \varepsilon$. Here, given f is continuous at $y = g(a)$, for given $\varepsilon > 0$, we have:

$$\exists \delta_f > 0\ \forall y \in \mathbb{R}\, (|y - g(a)| < \delta_f\ \to\ |f(y) - f(g(a))| < \varepsilon). \tag{3}$$

But, g is continuous at $x = a$ implying that for the found $\delta_f > 0$, in statement (3) we have:

$$\exists \delta_g > 0\ \forall x \in \mathbb{R}\, (|x - a| < \delta_g\ \to\ |g(x) -\ g(a)\,| < \delta_f). \tag{4}$$

Hence, taking $y = g(x)$, combining the statements (3) and (4) implies:

$$\exists \delta_g > 0\ \forall x \in \mathbb{R}\, (|x - a| < \delta_g\ \to\ |f(g(x)) - f(g(a))| < \varepsilon), \tag{5}$$

completing the proof.

Step(iv): First of all, we want for assigned $a > 0$ and given $\varepsilon > 0$, to find a $\delta > 0$ such that for all $x \in \mathbb{R}$ if $|x - a| < \delta$ then $|x^x - a^a| < \epsilon$. Here, without loss of generality, we may assume $0 < \epsilon < 1$.

Second, let $a > 0$ and consider the statement (2) with assumption $b = a\,ln(a)$, and $y = x\,ln(x)$ in which the exponential function is continuous. We consider the functions $f(x) = exp(x)$, $g(x) = x\,ln(x)$, and the composition function $f \circ g(x) = x^x$.

Third, since $|x^x - a^a| = |e^{xln(x)} - e^{aln(a)}|$, by statement (2), there is $\delta_* = ln(1 + \varepsilon e^{-aln(a)}) > 0$ such that:

$$\forall x \in \mathbb{R}\ (|xln(x) - aln(a)| < \delta_* \ \rightarrow\ |x^x - a^a| = |e^{xln(x)} - e^{aln(a)}| < \varepsilon). \tag{6}$$

Fourth, for the given $\delta_* > 0$, consider the inequality:

$$|xln(x) - aln(a)| \ \leq\ |x|\,|ln(x) - ln(a)| + |x - a||ln(a)|. \tag{7}$$

Sixth, take $\delta_{**} = \frac{\delta_*}{2|ln(a)|} > 0$. Then, if $|x - a| < \delta_{**}$, we have $|x - a||ln(a)| < \frac{\delta_*}{2}$. Also, $|x| \leq |x - a| + |a| = \delta_{**} + |a|$. Consequently, using statement (1), take $\delta_{***} = a(1 - e^{\frac{-\delta_*}{2(\delta_{**}+|a|)}}) > 0$. Then, if $|x - a| < \delta_{***}$, we have $|x|\,|ln(x) - ln(a)| < \frac{\delta_*}{2}$.

Seventh, take

$$\delta = \delta(a, \varepsilon) = \min(\delta_{**}, \delta_{***}) = \min(\frac{\delta_*}{2|ln(a)|}, a(1 - e^{\frac{-\delta_*}{2(\delta_{**}+|a|)}})) = \min(\frac{ln(1+\varepsilon e^{-aln(a)})}{2|ln(a)|}, a(1 - e^{\frac{-ln(1+\varepsilon e^{-aln(a)})}{2((\frac{ln(1+\varepsilon e^{-aln(a)})}{2|ln(a)|})+|a|)}})) \tag{8}$$

Eight, given the definition of $\delta = \delta(a, \varepsilon) > 0$, we have:

$$\forall x \in \mathbb{R}\ (|x - a| < \delta \rightarrow |xln(x) - aln(a)| \ \leq |x|\,|ln(x) - ln(a)| + |x - a||ln(a)| < \frac{\delta_*}{2} + \frac{\delta_*}{2} = \delta_*). \tag{9}$$

Accordingly, by statements (6) and (9) we have:

$$\delta = \delta(a, \varepsilon) = \min(\frac{ln(1+\varepsilon e^{-aln(a)})}{2|ln(a)|}, a(1 - e^{\frac{-ln(1+\varepsilon e^{-aln(a)})}{2((\frac{ln(1+\varepsilon e^{-aln(a)})}{2|ln(a)|})+|a|)}})): \forall x \in \mathbb{R}\ (|x - a| < \delta \rightarrow |x^x - a^a| < \varepsilon).$$

This completes the solution. QED.

Dear Problem Solvers,

I really hope that you enjoyed and learned something new by solving problem 4 and problem 5, time to move on, shown below are the next two problems.

Problem 6

Proposed by Ivan Retamoso, BMCC, USA.

Given that m and n are real numbers, without solving the equation determine how many real roots the following equation has:

$$(x-m-n)(x-m)=1$$

Problem 7

Proposed by Ivan Retamoso, BMCC, USA.

A fence 8ft tall runs parallel to a tall building at a distance of 4ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?