The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Hello Problem Solvers, I got solutions to **Problem 6** and to **Problem 7** and I am happy to inform that they were correct, interesting, and ingenious. By posting what I considered to be the best solutions, I hope to enrich and enhance the mathematical knowledge of our international community.

Solutions to **Problems** from the Previous Issue

Interesting algebra problem.

Proposed by Ivan Retamoso from Borough of Manhattan Community College, City university of New York, USA

Problem 6

Given that m and n are real numbers, without solving the equation determine how many real roots the following equation has:

$$(x-m-n)(x-m)=1$$

Solution to problem 6

by Jesse Wolf, Borough of Manhattan Community College, City university of New York, USA.

This solution is concise and utilizes the property of the discriminant of a quadratic equation in relation to the number of real roots of the equation.

A quadratic equation has 2 (distinct) real roots when the discriminant $D = b^2 - 4ac$ is greater than 0.

$(x - m - n)(x - m) - 1 = 0.$

$(1)x^2 + (-2m - n)x + (m^2 + nm - 1) = 0.$

a, b, c are the respective quantities in parentheses.

$D = n^2 + 4 > 0$ for all real m & n.

So 2 distinct real roots for all $\boldsymbol{m}$ & $\boldsymbol{n}$.

Solution to problem 6

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

For the readers who like complete solutions without skipping steps, here is a solution showing all the details, it is also based on the property of the discriminant of a quadratic equation in relation to the number of real roots of the equation.

First notice that $(x - m - n)(x - m)$ -1

is a second degree polynomial in the variable x and hence it will have 2 zeros.

(by fundamental theorem of algebra).

Hence either both solutions are real or both solutions are complex.

(Since if z is a complex solution for a polynomial and it's conjugate will also be solution of the same polynomial).

Consider the given equation

$(x - m - n)(x - m) = 1$

$(x-m-n)(x-m)-1=0$

$x^2-xm-xn-xm+m^2+mn-1=0$

$x^2-x(2m+n)+m^2+mn-1=0$

Hence the discrimination is

$(2m+n)^2-4(m^2+mn-1)$

$=4m^2+n^2+4mn-4m^2-4mn+4$

$=n^2+4>0$

Hence the both roots of the above equation $(x-m-n)(x-m)=1$

are real.

Interesting applied optimization problem.

Problem 7

Proposed by Ivan Retamoso, BMCC, USA.

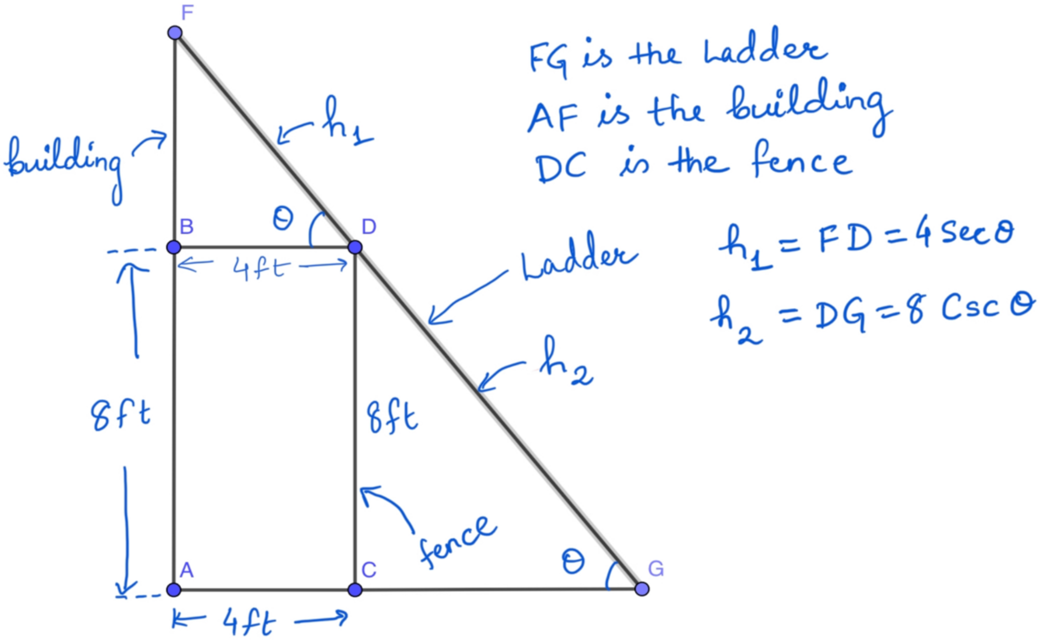
A fence 8ft tall runs parallel to a tall building at a distance of 4ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

Solution to problem 7

by Aradhana Kumari, Borough of Manhattan Community College, City university of New York, USA.

When solving Applied Optimization Problems in Calculus, often we underestimate the power of working with angles, this solution shows that a complicated problem becomes simple by expressing the objective function solely in terms of an angle, the second derivative test is used to identify the critical point as a minimum, the final solution is expressed in exact form.

As per question we have the below diagram



$h_1 = 4\sec\theta$

$h_2 = 8\csc\theta$

As per question, length of the ladder $=$ FG

$$= h_1 + h_2$$

$$= 4\sec\theta + 8\csc\theta$$

Let us assume $f(\theta) = 4\sec\theta + 8\csc\theta$

Then

$f'(\theta) = 4\sec\theta\tan\theta - 8\csc\theta\cot\theta$

For maxima or minima we equate the derivative of f to zero.

Hence, we have

$4\sec\theta\tan\theta - 8\csc\theta\cot\theta = 0$

$4(\sec\theta\tan\theta - 2\csc\theta\cot\theta) = 0$

$\sec\theta\tan\theta - 2\csc\theta\cot\theta = 0$

$$\left(\frac{1}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\right)-\left(\frac{2}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}\right)=0$$

$$\frac{\sin\theta}{(\cos\theta)^2}-\frac{2\cos\theta}{(\sin\theta)^2}=0$$

$$\frac{(\sin\theta)^3-2(\cos\theta)^3}{(\cos\theta)^2(\sin\theta)^2}=0$$

Therefore

$$(\sin\theta)^3-2(\cos\theta)^3=0$$

$$(\sin\theta)^3=2(\cos\theta)^3$$

$$(\tan\theta)^3=2$$

$$\tan\theta=\sqrt[3]{2}$$

Therefore

$$\theta=\tan^{-1}\left(\sqrt[3]{2}\right)$$

$$\sim 51.56°$$

Now we must check whether $\theta=\tan^{-1}\left(\sqrt[3]{2}\right) \quad \sim 51.56°$ is a maxima or minima.

For this we will find the second derivative of f.

Recall

$$\text{f}(\theta)=4\sec\theta+8\csc\theta$$

$$\text{f}'(\theta)=4\sec\theta\tan\theta-8\csc\theta\cot\theta=\left(\frac{4}{\cos\theta}\times\frac{\sin\theta}{\cos\theta}\right)-\left(\frac{8}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}\right)$$

$$\text{f}''(\theta)=\frac{[4\cos\theta\ (\cos\theta)^2)-\{4\sin\theta\times 2\times\cos\theta\times(-\sin\theta)\}]}{(\cos\theta)^4}-\frac{8\ [\{(-\sin\theta)\times(\sin\theta)^2\}-\{\cos\theta\times 2\sin\theta\times\cos\theta\}]}{(\sin\theta)^4}$$

$$=\frac{4(\cos\theta)^3+8(\sin\theta)^2\cos\theta}{(\cos\theta)^4}+\frac{8(\sin\theta)^3+16(\cos\theta)^2\sin\theta}{(\sin\theta)^4}$$

Since $\theta=\tan^{-1}\left(\sqrt[3]{2}\right) \quad \sim 51.56°$

the angle 51.56° is in the first quadrant. We know that both the $\sin\theta$ and $\cos\theta$ are positive in the first quadrant

Hence

f '' $(\theta) = \frac{4(\cos\theta)^4+8(\sin\theta)^2\cos\theta}{(\cos\theta)^4} + \frac{8(\sin\theta)^3+16(\cos\theta)^2\ \sin\theta}{(\sin\theta)^4}$ is positive.

Therefore $\theta = \tan^{-1}\left(\sqrt[3]{2}\right) \ \sim 51.56°$ is a point of minima for the function f.

Hence the length of the shortest ladder is $h_1 + h_2$

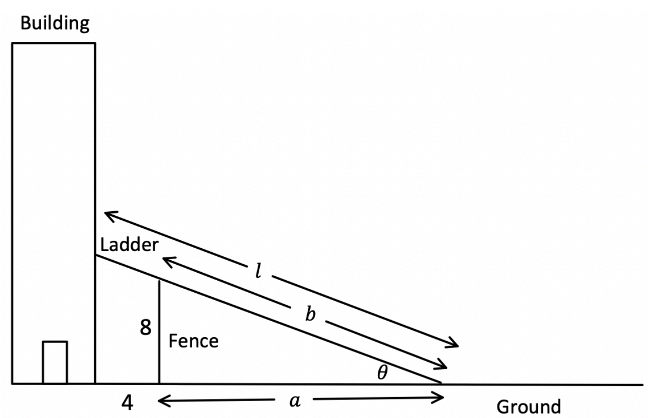
$= \mathbf{4\sec\theta + 8\csc\theta}$

$= \mathbf{4\sec\left(\tan^{-1}\left(\sqrt[3]{2}\right)\right) + 8\csc\left(\tan^{-1}\left(\sqrt[3]{2}\right)\right)}$

Solution to problem 7

by Ivan Retamoso (the proposer), Borough of Manhattan Community College, City university of New York, USA.

This solution confirms the advantage of choosing to work with angles to represent the function that needs to be minimized, for identifying the critical point as a minimum this solution uses the first derivative test, the final solution is computed as a decimal number rounded to two decimal places.



$\frac{a}{8} = cot\theta$ then $a = 8cot\theta$

$\frac{b}{8} = csc\theta$ then $a = 8csc\theta$

By similarity of right triangles

$$\frac{L}{8csc\theta} = \frac{4 + 8cot\theta}{8cot\theta}$$

Then

$L = 4sec\theta + 8csc\theta$

Setting $\frac{dL}{d\theta} = 0$ and solving for θ then

$4sec\theta tan\theta - 8csc\theta cot\theta = 0$

$4sec\theta tan\theta = 8csc\theta cot\theta$

$tan^3\theta = 2$

$tan\theta = \sqrt[3]{2}$

$\theta = \arctan(\sqrt[3]{2})$

$\theta = 51.56°$

Notice that $0° < \theta < 90°$ then $tan\theta$ is strictly increasing.

For $0° < \theta < 51.56°$ then $\frac{dL}{d\theta} = 4sec\theta tan\theta - 8csc\theta cot\theta < 0$

For $51.56° < \theta < 90°$ then $\frac{dL}{d\theta} = 4sec\theta tan\theta - 8csc\theta cot\theta > 0$

Then L achieves its minimum at $\theta = 51.56°$
Then $L_{min} = 4sec51.56° + 8\csc 51.56°$

Then $L_{min} = 16.65\ ft.$

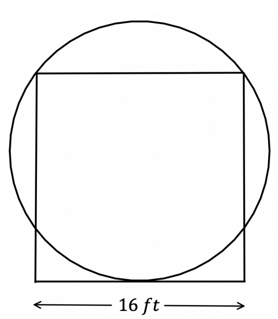
Dear Problem Solvers,

I really hope that you enjoyed and more importantly learned something new by solving problem 6 and problem 7, time to move forward so below are the next two problems.

Problem 8

Proposed by Ivan Retamoso, BMCC, USA.

In the figure below find the ratio between the perimeter of the circle (circumference) and the perimeter of the square in exact form.



Problem 9

Proposed by Ivan Retamoso, BMCC, USA.

It is needed to construct a rain gutter from a metal sheet of width 36 cm by bending up one-third of the sheet on each side through an angle α. Find the value of α such that the gutter will carry the maximum amount of water.