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The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their "comfort zone" and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I am delighted to announce that I have received solutions to both Problem 10 and Problem 11, and I am pleased to report that they were all correct, as well as fascinating and innovative. By showcasing what I deemed to be the most outstanding solutions, I aim to enrich and elevate the mathematical understanding of our global community.

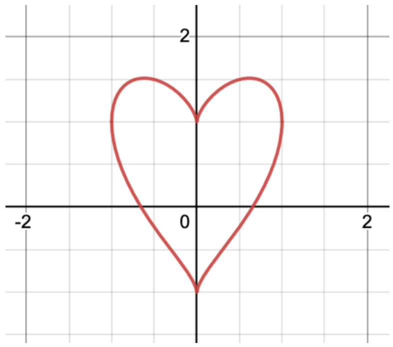
Solutions to **Problems** from the Previous Issue

"Heart" problem.

Problem 10

Proposed by Ivan Retamoso, BMCC, USA.

The Graph of the equation $\left(y - x^{\frac{2}{3}}\right)^2 = 1 - x^2$ is a "heart" and is shown below:



Find the slope of the tangent line of the "heart" at the point $\left(\frac{1}{8}, \frac{2+3\sqrt{7}}{8}\right)$ in exact form.

Solution to problem 10

By Masayuki Kirsch, Borough of Manhattan Community College, Japan.

This minimalist and efficient solution takes full advantage of the local nature of the slope of the tangent line, rather than computing $\frac{dy}{dx}$ *in general the substitution of the values of the coordinates* $\left(\frac{1}{8}, \frac{2+3\sqrt{7}}{8}\right)$ *is done immediately after taking derivatives of both sides of the equation, which quickly leads to the computation of the slope in exact form, kudos to this great solution.*

$$\frac{d}{dx}\left(y - x^{\frac{2}{3}}\right)^2 = \frac{d}{dx}\left(1 - x^2\right)$$

$$2\left(y - x^{\frac{2}{3}}\right)\left(\frac{dy}{dx} - \frac{2}{3x^{\frac{1}{3}}}\right) = -2x$$

Substitution =

$$2\left(\frac{2+3\sqrt{7}}{8} - \frac{1}{8}^{\frac{2}{3}}\right)\left(m - \frac{2}{3\left(\frac{1}{8}\right)^{\frac{1}{3}}}\right) = -2\left(\frac{1}{8}\right)$$

$$= \frac{3\sqrt{7}}{4}\left(m - \frac{4}{3}\right)$$

$$= \frac{3\sqrt{7}\,m}{4} - \sqrt{7} = -\frac{1}{4}$$

$$\left[m = -\frac{\sqrt{7}-28}{21}\right]$$

Second Solution to problem 10

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This second solution follows a different approach, first $\frac{dy}{dx}$ *is symbolically obtained as a general formula, then the final solution if found by applying the general formula for a particular case.*

Consider the equation

$$\left(y-x^{\frac{2}{3}}\right)^2 = 1-x^2$$

differentiate both side with respect to x we get

$$\frac{d}{dx}\left(y-x^{\frac{2}{3}}\right)^2 = \frac{d}{dx}(1-x^2)$$

$$2\left(y-x^{2/3}\right)\left\{\frac{d}{dx}\left[y-x^{2/3}\right]\right\} = -2x$$

$$2\left(y-x^{2/3}\right)\left\{\left[\frac{dy}{dx}-\frac{2}{3}x^{-\frac{1}{3}}\right]\right\} = -2x$$

$$\frac{dy}{dx}-\frac{2}{3}x^{-\frac{1}{3}} = \frac{-2x}{2\left(y-x^{\frac{2}{3}}\right)}$$

$$\therefore \frac{dy}{dx} = \frac{-2x}{2\left(y-x^{2/3}\right)}+\frac{2}{3}\frac{1}{x^{1/3}}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{8},\frac{2+3\sqrt{7}}{8}\right)} = \frac{-2\times\frac{1}{8}}{2\left[\frac{2+3\sqrt{7}}{8}-\left(\frac{1}{8}\right)^{2/3}\right]}+\frac{2}{3}\frac{1}{\left(\frac{1}{8}\right)^{1/3}}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{1}{8},\frac{2+3\sqrt{7}}{8}\right)} = \frac{-\frac{1}{4}}{\frac{2+3\sqrt{7}}{4}-2\times\frac{1}{4}}+\frac{2}{3}\frac{1}{\left(\frac{1}{2}\right)}$$

$$= \frac{-\frac{1}{4}}{\frac{2+3\sqrt{7}-2}{4}}+\frac{4}{3}$$

$$= \left(\frac{-1}{4}\times\frac{4}{3\sqrt{7}}\right)+\frac{4}{3}$$

$$= -\frac{1}{3\sqrt{7}}+\frac{4}{3} = \frac{-1}{3}\left[\frac{1}{\sqrt{7}}-4\right]$$

$$= \frac{-1}{3}\left[\frac{\sqrt{7}}{7}-4\right] = \frac{-1}{3}\left[\frac{\sqrt{7}-28}{7}\right]$$

$$= \frac{28-\sqrt{7}}{21}$$

$\therefore$ The slope of the tangent line to the graph of the equation $\left(y-x^{\frac{2}{3}}\right)^2 = 1-x^2$ at the point $\left(\frac{1}{8},\frac{2+3\sqrt{7}}{8}\right)$ is $\frac{28-\sqrt{7}}{21}$

Interesting “Parallel lines” problem.

Problem 11

Proposed by Ivan Retamoso, BMCC, USA.

Find the coordinates of the point (x, y) that belongs to the graph of $x^2 + 18xy + 81y^2 = 144$ that is closest to the origin $(0,0)$ and lies on the third quadrant.

Solution to problem 11

By Dr. Michael W. Ecker, (retired) Pennsylvania State University, USA.

We are truly privileged to have the distinguished Dr. Michael W. Ecker, a prominent figure in the world of problem-solving, provide his outstanding solution to problem 11. Dr. Ecker's remarkable achievements include a PhD in mathematics from the City University of New York in 1978, and an illustrious 45-year career as a mathematics professor, with the last 30 years spent at Pennsylvania State University's Wilkes-Barre campus. In addition to his numerous academic accolades, Dr. Ecker also serves as the Problem Section Editor of the MathAMATYC Educator journal. It is an honor to receive his valuable contribution.

Solution: The left side is a perfect square so the equation is equivalent to (*) $(x+9y)^2 = 12^2$. Note that replacing x by $-x$ *and also* y by $-y$ in (*) results in an equivalent equation. That means that the graph is symmetric with respect to the origin, which we will call O.

Next, by taking square roots, the graph of (*) is equivalent to the point (x, y) lying on the graph of one of the two lines with equations $x+9y=12$ and $x+9y=-12$. The second of these is in the third quadrant. Let's call the closest point on this line (to the origin) A. Then the line segment OA with this shortest length is found by dropping a perpendicular from the origin to the line.

Rewrite $x+9y=-12$ as $y=-\frac{x}{9}-\frac{4}{3}$ to identify our original line as having slope = $-\frac{1}{9}$. Hence, our perpendicular line has slope equal to the negative reciprocal of this, or +9. It follows that the equation of our perpendicular, by the Point-Slope formula, is $y=9x$. So, our nearest point lies on both the lines $y=-\frac{x}{9}-\frac{4}{3}$ and $y=9x$. To find the intersection, we set $9x=-\frac{x}{9}-\frac{4}{3}$ and solve in usual fashion, such as multiplying each side by 9 and isolating x. So, we get $x=-\frac{6}{41}$, and substituting this back into either of the two equations gives $y=-\frac{54}{41}$. It is straightforward to check, at least, that $\left(-\frac{6}{41},-\frac{54}{41}\right)$ does satisfy the linear equation $x+9y=-12$.

Second Solution to problem 11

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This great second solution is detailed oriented and includes a graph to support the steps that were followed to find the solution.

Consider the equation

$x^2+18xy+81y^2 = 144$

$(x+9y)^2 = 12^2$

$(x+9y)^2-12^2 = 0$

$(x+9y-12)(x+9y+12) = 0$

$\Rightarrow x+9y-12=0$ or $x+9y+12=0$

$\Rightarrow x+9y=12$ or $x+9y=-12$

$\Rightarrow y = -\frac{x}{9}+\frac{12}{9}$ or $y=-\frac{x}{9}-\frac{12}{9}$

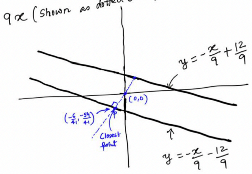
These represents set of parallel lines with slope $-\frac{1}{9}$.

In order to find the minimum distance between a point (x,y) lying on the equation $x^2+18xy+81y^2=144$ (which represents a set of parallel lines) and the point $(0,0)$.

First we need to find the equation of perpendicular line to the line $y = -\frac{x}{9}+\frac{12}{9}$ & which passes through $(0,0)$.

$\therefore$ The equation of perpendicular line to $y=-\frac{x}{9}+\frac{12}{9}$ which passes through $(0,0)$ is

$y = 9x$ (shown as dotted blue)



Point $P(x,y)$ is the intersection point between the line $y=-\frac{x}{9}-\frac{12}{9}$ & $y=9x$.

We need to find the coordinate of the point $P(x,y)$. For this consider

$y = -\frac{x}{9}-\frac{12}{9}$

$9x = -\frac{x}{9}-\frac{12}{9}$ (for intersection point)

$9x+\frac{x}{9} = -\frac{12}{9}$

$\frac{82x}{9} = \frac{-12}{9} \Rightarrow 82x=-12$

$x = \frac{-12}{82} = \frac{-6}{41}$

$\therefore y = -9\times\frac{6}{41} = \frac{-54}{41}$

Hence the coordinate of P is $\left(\frac{-6}{41},\frac{-54}{41}\right)$

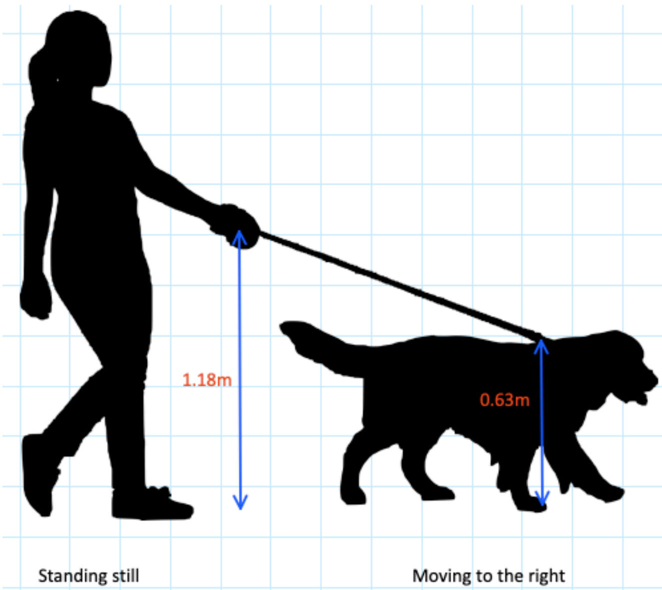
Dear fellow problem solvers,

I trust that you not only enjoyed but also gained valuable insights from solving problems 10 and 11. It's time to move on to the next challenges, and I am excited to present you with the following two problems.

Problem 12

Proposed by Ivan Retamoso, BMCC, USA.

Eva is standing still holding her dog via an extendable leash which she keeps at the heigh of 1.18 m above the ground as shown in the figure below, suddenly her dog walks to the right at a constant speed of 0.9 $\frac{m}{s}$, at what rate is the leash extending when the end of the leash is $3m$ horizontally away from Eva?



Problem 13

Proposed by Ivan Retamoso, BMCC, USA.

Prove that the equation $x^3 - 14x + k = 0$ where k is any real number, has at most one real number solution in the interval $[-2,2]$.