The Problem Corner

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The Purpose of **The Problem Corner** is to give Students and Instructors working independently or together a chance to step out of their “comfort zone” and solve challenging problems. Rather than in the solutions alone, we are interested in methods, strategies, and original ideas following the path toward figuring out the final solutions. We also encourage our Readers to propose new problems. To submit a solution, type it in Microsoft Word, using math type or equation editor, however PDF files are also acceptable. Email your solution as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country. Solutions to posted problem must contain detailed explanation of how the problem was solved. The best solution will be published in a future issue of MTRJ, and correct solutions will be given recognition. To propose a problem, type it in Microsoft Word, using math type or equation editor, email your proposed problem as an attachment to The Problem Corner Editor iretamoso@bmcc.cuny.edu stating your name, institutional affiliation, city, state, and country.

Greetings, fellow problem solvers!

I am delighted to announce that I have received solutions to both Problem 12 and Problem 13, and I am pleased to report that they were all correct, as well as fascinating and innovative. By showcasing what I deemed to be the most outstanding solutions, I aim to enrich and elevate the mathematical understanding of our global community.

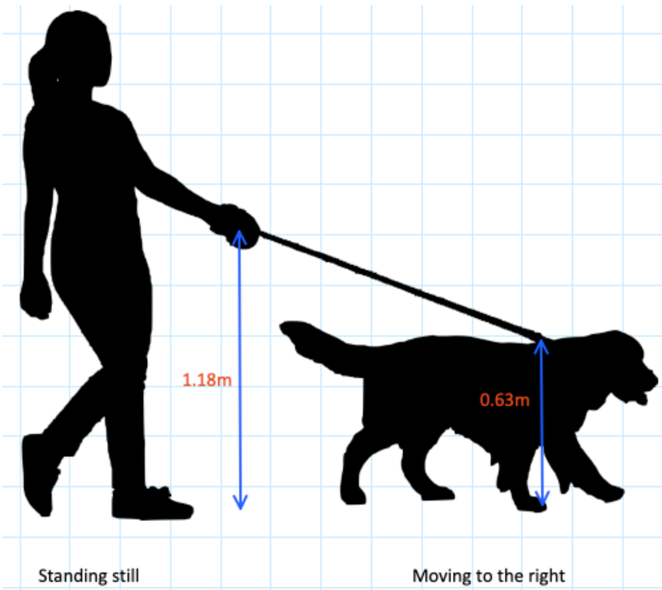
Solutions to **Problems** from the Previous Issue

Interesting “Dog walking” problem.

Problem 12

Proposed by Ivan Retamoso, BMCC, USA.

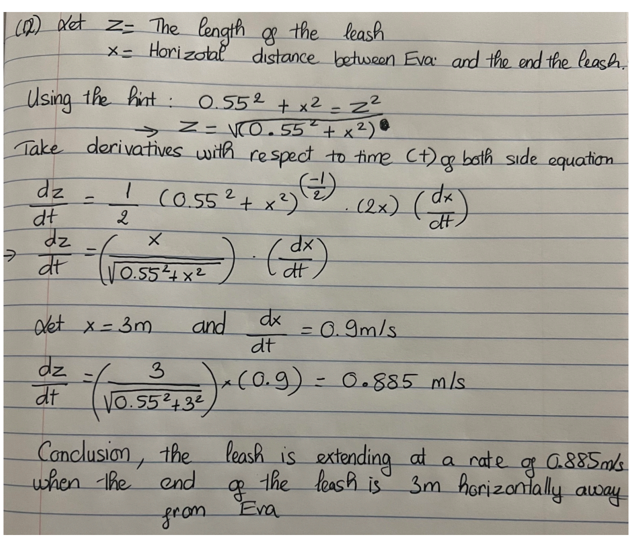
Eva is standing still holding her dog via an extendable leash which she keeps at the heigh of $1.18\ m$ above the ground as shown in the figure below, suddenly her dog walks to the right at a constant speed of $0.9\,\frac{m}{s}$, at what rate is the leash extending when the end of the leash is $3m$ horizontally away from Eva?



Solution to problem 12

By Phuong Uy Nguyen, Borough of Manhattan Community College, Vietnam.

This efficient solution employs carefully chosen variables to establish an equation based on the Pythagorean theorem, ensuring its validity throughout Eva and her dog's motion. By isolating the main variable and utilizing differentiation with respect to time, incorporating the chain rule, our problem solver determines the rate at which the leash extends for the specified distance.



(Q) Let z = The length of the leash
x = Horizotal distance between Eva and the end the leash.

Using the hint: $0.55^2 + x^2 = z^2$

$\rightarrow z = \sqrt{(0.55^2 + x^2)}$

Take derivatives with respect to time (t) of both side equation

$$\frac{dz}{dt} = \frac{1}{2}(0.55^2 + x^2)^{\left(\frac{-1}{2}\right)} \cdot (2x)\left(\frac{dx}{dt}\right)$$

$$\Rightarrow \frac{dz}{dt} = \left(\frac{x}{\sqrt{0.55^2 + x^2}}\right) \cdot \left(\frac{dx}{dt}\right)$$

Let $x = 3m$ and $\frac{dx}{dt} = 0.9 m/s$

$$\frac{dz}{dt} = \left(\frac{3}{\sqrt{0.55^2 + 3^2}}\right) \times (0.9) = 0.885 \ m/s$$

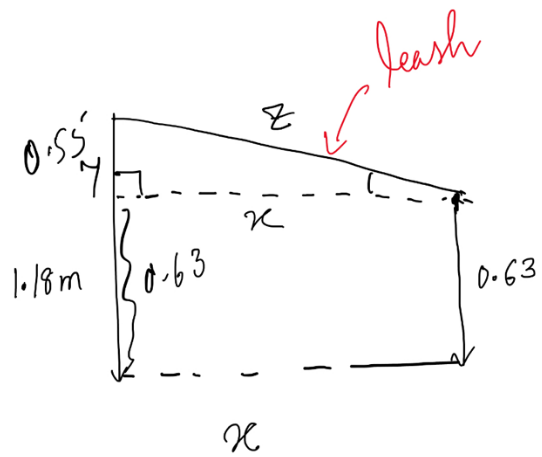
Conclusion, the leash is extending at a rate of 0.885m/s when the end of the leash is 3m horizontally away from Eva

Second Solution to problem 12

By Aradhana Kumari, Borough of Manhattan Community College, USA.

This second solution adopts a different approach by employing implicit differentiation. After finding the derivative with respect to time for both sides of the equation, which holds true throughout Eva and her dog's motion, the required rate of change is determined. Each step of the solution is thoroughly justified, ensuring clarity and accuracy. Furthermore, a diagram is included to enhance visualization and provide a clearer understanding of the problem.

Solution: As per question we have the below diagram



From the above diagram

When x=3, y= (1.18-0.63= 0.55) we have

$x^2 + y^2 = z^2$..............(1)

$3^2 + (0.55)^2 = z^2$

$9 + 0.3025 = z^2$

Hence $z = \sqrt{(9.3025)}$

$z = 3.05$

Differentiate the equation given by (1) with respect to time we get

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

x $\frac{dx}{dt}$ + y $\frac{dy}{dt}$ = z $\frac{dz}{dt}$ ....(2)

Substituting the value x=3, $\frac{dx}{dt}$ = .9, z = 3.05, and $\frac{dy}{dt}$ = 0 (Since the dog is moving with the constant speed on the x-direction) in equation in given by (2) we get

$3\times(0.9) = 3.05\frac{dz}{dt}$

$\frac{dz}{dt} = \frac{3\times(0.9)}{3.05} = .88524590163 \approx .89$ m/s

Hence the rate at which the lease extending when the end of the lease is 3m horizontally away from the Eva is approximately .89 m/s.

"Intermediate Value Theorem" problem.

Problem 13

Proposed by Ivan Retamoso, BMCC, USA.

Prove that the equation $x^3 - 14x + k = 0$ where k is any real number, has at most one real number solution in the interval $[-2,2]$.

Solution to problem 13

By Jesse Wolf, Borough of Manhattan Community College, USA.

This comprehensive step-by-step solution utilizes various mathematical tools to ensure a thorough analysis. By incorporating the first derivative, the second derivative test, and The Intermediate Value Theorem, this solution covers all possible cases and provides rigorous justifications for each step.

y = x^3 - 14x + k

y' = 3x^2 -14

y" = 6x

y'= 0 => x = + or - ((14/3)^.5) = ~ (+ or - 2.2).

y"(-(14/3)^.5) < 0; y"((14/3)^.5) > 0.

Second Derivative Test => there exists a relative max at x = -(14/3)^.5 and a relative min at x = (14/3)^.5.

So, y(-(14/3)^.5) > y((14/3)^.5) and is strictly decreasing on (-(14/3)^.5,(14/3)^.5)) which contains [-2,2].

1:

If y(-(14/3)^.5) > 0 and y((14/3)^.5) > or = 0 => there exists 0 zeros of y on (-(14/3)^.5,(14/3)^.5)) and thus 0 zeros on [-2,2].

2:

If y(-(14/3)^.5) > 0 and y((14/3)^.5) < 0 => there exists at most one zero of y on [-2,2]:

2a:

The fact that y is continuous and strictly decreasing on

(-(14/3)^.5,(14/3)^.5)) => (via the Intermediate Value Theorem) that there exists a unique zero on that interval.

If the zero occurs on (-(14/3)^.5, -2) or (2, (14/3)^.5)) then there exist 0 zeros on [-2,2].

2b:

If the zero occurs on [-2,2] there exists 1 zero on [-2,2].

3:

If y(-(14/3)^.5) = 0 and y((14/3)^.5) < 0 => there exists 0 zeros of y on (-(14/3)^.5,(14/3)^.5)) and thus 0 zeros on [-2,2].

4:

If y((-(14/3)^.5)) < 0 and y((14/3)^.5)) < 0 => there exists no zeros of y on (-(14/3)^.5,(14/3)^.5)) and thus 0 zeros on [-2,2].

QED

Second Solution to problem 13

By Ivan Retamoso (proposer), Borough of Manhattan Community College, USA.

This alternative solution employs a distinct methodology. It initiates by establishing the negativity of the derivative of the left side of the equation within the given interval. Consequently, it deduces that the left side of the equation exhibits strict monotonicity, specifically, it must be strictly decreasing over the provided interval.

Let $f(x) = x^3 - 14x + k$

Then $f'(x) = 3x^2 - 14$

For all x in $[-2,2]$

$-2 \leq x \leq 2$

$|x| \leq 2$

$|x|^2 \leq 2^2$

$x^2 \leq 4$

$3x^2 \leq 12$

$3x^2 - 14 \leq -2 < 0$

Then $f'(x) < 0$ for all x in $[-2,2]$

Then $f(x)$ is strictly decreasing on $[-2,2]$, since $f(x)$ is continuous then it means that the graph of $f(x)$ will intersect the x axis at most once.

Therefore, the equation $x^3 - 14x + k = 0$ where k is any real number, has at most one real number solution in the interval $[-2,2]$.

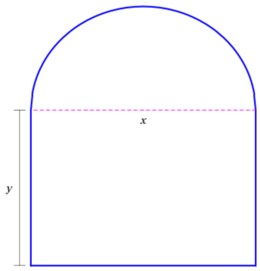
Dear fellow problem solvers,

I have confidence that your experience in solving problems 12 and 13 was not only enjoyable but also resulted in valuable insights. Now, let's progress to the next set of challenges, as I am genuinely thrilled to present you with the following two problems.

Problem 14

Proposed by Ivan Retamoso, BMCC, USA.

Let's imagine a scenario where a corral is being enclosed using 130 ft of fencing. The corral is in the shape of a rectangle, and it has a semicircle attached to one of its sides. The diameter of the semicircle aligns with the length of the rectangle, as depicted in the figure provided.



Determine the values of x and y that will result in the corral having the largest possible area.

Problem 15

Proposed by Ivan Retamoso, BMCC, USA.

x, y, and z are real numbers such that $x + y + z = 17$ and $\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} = \frac{4}{15}$ find the exact value of $\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$.