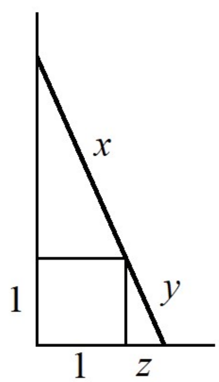
Solution II by Henry Ricardo, Westchester Area Math Circle, Purchase, NY. This problem belongs to a time-honored class of puzzles, often posed in terms of grazing goats or other animals. In his note "The Bull and the Silo: An Application of Curvature" [*Amer. Math. Monthly* **105**(1998), 55-58], Michael E. Hoffman refers to the known answer if the chain has length L and the structure to which the animal is tethered has a circular cross section of radius R: $A = \frac{\pi L^2}{2} + \frac{L^3}{3R}$ if $L \le R\pi$. In our problem, $L = 50\pi$, $R = 50$, so the total area is $A = \frac{\pi(50\pi)^2}{2} + \frac{(50\pi)^2}{3(50)} = \frac{5}{6}(50)^2\pi^3$ $= \frac{6250\pi}{3}$ square feet.

Two incorrect solutions were also received.

Ladder Leanings

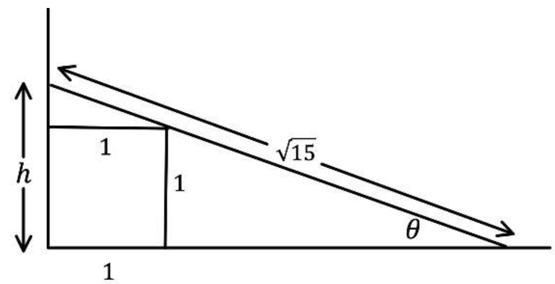
I-2 Borrowed by Michael W. Ecker. A unit cube is placed adjacently to a wall that is at a right angle to the ground. An idealized ladder with length $\sqrt{15}$ leans against the cube and the wall. How high does the ladder reach to the top of the wall?

Solution I by Raymond N. Greenwell (Emeritus), Hofstra University, Hempstead, NY. Let x, y, and z be as in the drawing, where $x + y = \sqrt{15}$ is the length of the ladder.



Notice by similar triangles that $\frac{z}{y} = \frac{1}{x}$, so now $z = \frac{y}{x}$. By the Pythagorean Theorem $1 + z^2 = y^2$, implying $1 + \frac{y^2}{x^2} = y^2$, and then $x^2 + y^2 = x^2y^2$, from which $x^2 + \left(\sqrt{15} - x\right)^2 =$ $x^2\left(\sqrt{15} - x\right)^2$. Then multiplying out and simplifying yields $2x^2 - 2\sqrt{15}\,x + 15 = x^4 - 2\sqrt{15}\,x^3 + 15x^2$. As a result, $x^4 - 2\sqrt{15}\,x^3 + 13x^2 + 2\sqrt{15}\,x - 15 = 0$. We factor to get $(x^2 - \sqrt{15}\,x + 3)(x^2 - \sqrt{15}\,x - 5) = 0$, and using the quadratic formula twice we find four roots, only two of which lie between 1 and $\sqrt{15}$: $\frac{\sqrt{15}+\sqrt{3}}{2}$ and $\frac{\sqrt{15}-\sqrt{3}}{2}$. Finally, the Pythagorean Theorem tells us that the point where the ladder hits the wall is at $1 + \sqrt{x^2 - 1}$. Then substituting and simplifying gives possible heights of $\frac{5+\sqrt{5}}{2} \approx 3.62$ and $\frac{5-\sqrt{5}}{2} \approx 1.38$.

Solution II by Ivan Retamoso. Let h be how high the ladder reaches up the wall, and let θ be the angle formed by the ladder and the ground as shown in the figure below. The triangle on the top of the cube is similar to the triangle on the right of the cube.



Then

$$\frac{\sqrt{15}\sin\theta - 1}{1} = \frac{1}{\sqrt{15}\cos\theta - 1},$$

so $15\sin\theta\cos\theta = \sqrt{15}\sin\theta + \sqrt{15}\cos\theta$. Notice that $0 < \theta < \frac{\pi}{2}$, guaranteeing $\sin\theta$, $\cos\theta$, and $\sin\theta\cos\theta$ are positive, so we divide by the radical and square both sides of the equation to get

$$15\sin^2\theta\cos^2\theta = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta.$$

Simplifying gives $15\sin^2\theta\cos^2\theta - 2\sin\theta\cos\theta - 1 = 0$ or, after factoring, $(3\sin\theta\cos\theta - 1)(5\sin\theta\cos\theta + 1) = 0$. So $\sin\theta\cos\theta = \frac{1}{3}$ but not $\frac{-1}{5}$, since it was guaranteed to be positive. Doubling and using the inverse sine, we get $2\theta = \arcsin\left(\frac{2}{3}\right)$ or $2\theta = \pi - \arcsin\left(\frac{2}{3}\right)$, because $0 < \theta < \frac{\pi}{2}$ says $0 < 2\theta < \pi$. Therefore $\theta = \frac{\arcsin\left(\frac{2}{3}\right)}{2}$ or $\theta = \frac{\pi - \arcsin\left(\frac{2}{3}\right)}{2}$; and since $h = \sqrt{15}\sin\theta$, we obtain $h = \frac{5}{2} - \frac{\sqrt{5}}{2} \approx 1.38$ or $h = \frac{5}{2} + \frac{\sqrt{5}}{2} \approx 3.62$.

Also solved by Austin Jones, Wake Technical Community College, Raleigh, NC; and Troy D. Williamson, Texas State Technical College, Abilene, TX.

m-gon "Scallops"

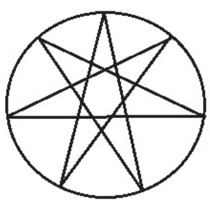
I-3 Proposed by Stephen L. Plett. Let $m > 2$ be an integer. In a circle of radius m, inscribe a regular m-gon. a) Find S_m, the total area inside the circle but outside the m-gon. b) $\lim_{m\to\infty} S_m = ?$

Composite of solutions by Wei-Lai, University of South Carolina, Salkehatchie, Walterboro, SC; and Mark Moodie, San Jacinto College, Houston, TX. For (a), from the center we dissect the m-gon into m congruent isosceles triangles having central angle $\theta = \frac{2\pi}{m}$ and sides of length m. Using trigonometry we find the area of each triangle to be $m^2 \sin\left(\frac{\pi}{m}\right)\cos\left(\frac{\pi}{m}\right) = \frac{m^2}{2}\sin\frac{2\pi}{m}$. This means the area of the m-gon is $\frac{m^3}{2}\sin\frac{2\pi}{m}$, and the desired area is just $S_m = \pi m^2 - \frac{m^3}{2}\sin\frac{2\pi}{m}$ or $S_m = \frac{8\pi^4 m^{-1} - 4\pi^3 \sin\left(2\pi m^{-1}\right)}{8\pi^3 m^{-3}}$. For (b), $\lim_{m\to\infty} S_m = \lim_{m\to\infty} \frac{4\pi^3\left(2\pi m^{-1}\right) - 4\pi^3 \sin\left(2\pi m^{-1}\right)}{\left(2\pi m^{-1}\right)^3}$ and substituting $x = 2\pi m^{-1}$, we then have $\lim_{m\to\infty} S_m = \lim_{x\to 0} \frac{4\pi^3(x - \sin x)}{x^3}$ $\overset{L'H}{=} \lim_{x\to 0} \frac{4\pi^3(1-\cos x)}{3x^2} \overset{L'H}{=} \lim_{x\to 0} \frac{4\pi^3(\sin x)}{6x} = \frac{2\pi^3}{3}$.

Also solved by Austin Jones; Ivan Retamoso; Henry Ricardo; Troy Williamson; and the proposer.

Shrinking *n*-grams

I-4 Proposed by Stephen L. Plett. Let $i > 1$ be an integer, and let $n = 2i + 1$, $r = \frac{1}{i}$. Place n equally-spaced points on a circle of radius r, labeled sequentially $P_1, P_2, P_3, \ldots, P_n$. Now form the *r-radius n-gram* by drawing line segments: $P_1P_{i+1}, P_{i+1}P_{2i+1}, P_{2i+1}P_i, \ldots, P_{i+2}P_1$. (Note: When $i = 2$, then $n = 5$ and we have a pentagram.) Using L_n = the sum of the lengths of the n line segments, find $\lim_{n\to\infty} L_n$.



The case $n = 7$

Composite of solutions by Austin Jones and the proposer. Since $n = 2i + 1$, $i = \frac{n-1}{2}$ and $r = \frac{2}{n-1}$. We draw radii to each vertex to create n central angles of $\frac{2\pi}{n}$. Each chord P_1P_{i+1} subtends i of these angles, with corresponding angle $\theta = \frac{i(2\pi)}{n} = \frac{\pi(n-1)}{n}$. The chord's length, c, is then $c = 2r\sin\left(\frac{\theta}{2}\right) = \frac{4}{n-1}\cdot\sin\left(\frac{\pi(n-1)}{2n}\right)$. So

$$L_n = n \cdot c = \frac{4\sin\left(\frac{\pi}{2}\cdot\frac{n-1}{n}\right)}{\left(\frac{n-1}{n}\right)} = 4 \cdot \frac{\sin\left(\frac{\pi}{2}\cdot\left(1-\frac{1}{n}\right)\right)}{\left(1-\frac{1}{n}\right)}$$

and

$$\lim_{n\to\infty} L_n = 4 \cdot \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{2}\cdot\left(1-\frac{1}{n}\right)\right)}{\left(1-\frac{1}{n}\right)} = 4.$$

Also solved by Troy D. Williamson.

Poetry in Motion

I-5 Proposed by Stephen L. Plett. The following game is played on $F = \{(m,n) \mid m, n \text{ non-negative integers}\}$ (that is, on lattice points in the first quarter-plane). Start with velocity $\vec{v}_0 = \langle 0,0\rangle$ and position $\vec{r}_0 = \langle 0,0\rangle$. For each turn, the player can add one to either, neither, or both components of the velocity. The new position is calculated by adding the new velocity vector to the old position vector; i.e., $\vec{r}_{k+1} = \vec{v}_{k+1} + \vec{r}_k$. (Example: $\vec{v}_0 = \langle 0,0\rangle$, $\vec{r}_0 = \langle 0,0\rangle$; $\vec{v}_1 = \langle 1,0\rangle$, $\vec{r}_1 = \langle 1,0\rangle$; $\vec{v}_2 = \langle 2,1\rangle$, $\vec{r}_2 = \langle 3,1\rangle$; $\vec{v}_3 = \langle 2,2\rangle$, $\vec{r}_3 = \langle 5,3\rangle$; etc.) Let S_n be the set of all points of F that can be reached in at most n turns. Find, with any needed proofs of claims, a formula for $|S_n|$, the number of elements of S_n.

Essentially identical solutions by Troy D. Williamson and Austin Jones. Note that the game is symmetric with respect to coordinates, so $(a,b) \in S_n \Rightarrow (b,a) \in S_n$. Let P_n be the equivalent set in the one-dimensional game, so that $S_n = (P_n)^2$. Note a value $v \in P_n$ if there is partition of v into a sum of n non-negative numbers $v = v_1 + v_2 + \ldots + v_n$ where v_1 is 0 or 1, and $v_i - v_{i-1}$ is 0 or 1 for all $i = 2, 3, \cdots, n$. Thus, $\max|P_n| = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ and since $0 \in P_n$, then $|P_n| \leq \frac{n(n+1)}{2} + 1$. We now claim $P_n = \left\{x \mid 0 \leq x \leq \frac{n(n+1)}{2}\right\}$, so that $|S_m| = \frac{m(m+1)}{2} + 1$. Base case $n = 1$: $v_1 = 0$ means $0 \in P_n$ while $v_1 = 1$ means $1 \in P_n$. Inductive step: Assume $P_k = \left\{x \mid 0 \leq x \leq \frac{k(k+1)}{2}\right\}$ for some $k > 1$ and note that $P_k \subset P_{k+1}$ using $v_1 = 0$. For values $\frac{k(k+1)}{2} < x \leq \frac{(k+1)(k+2)}{2} = \frac{k(k+1)}{2} + k + 1$, start with the sum $\frac{k(k+1)}{2} = 1 + 2 + \ldots + k$ but making one $v_k = 0$ we can insert an appropriate value (e.g., $1 + 1 + 2 + 3 + \cdots + k$ or $1 + 2 + 2 + 3 + \cdots + k$, etc.) to achieve $\frac{k(k+1)}{2} + 1$, $\frac{k(k+1)}{2} + 2$, $\frac{k(k+1)}{2} + 3$, $\cdots$, $\frac{k(k+1)}{2} + k + 1$. Thus, $|P_n| = \frac{n(n+1)}{2} + 1$, and $|S_n| = \left(\frac{n(n+1)}{2} + 1\right)^2$.

Also solved by the proposer. One incorrect solution was also received.