# The Problem Section

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The problem section seeks lively and interesting mathematics problems and their solutions. *To propose a problem or offer a solution, prepare it as a Microsoft Word file, preferably saved in the older .doc format. Use Math Type or Equation Editor as needed for layout.*

<u>NEW POLICY</u>: *Please send new proposals to the Problem Section Editor* at DrMWEcker@aol.com along with your name, institutional affiliation, and city/state. Likewise, please send all proposed solutions along with your name, affiliation, and location by email to ***both*** the Solutions Editor at Prof.S.Plett@gmail.com ***and*** the Problem Section Editor at DrMWEcker@aol.com.

## Quicker Problems (Q) and Students' Quicker Challenges (SQ)

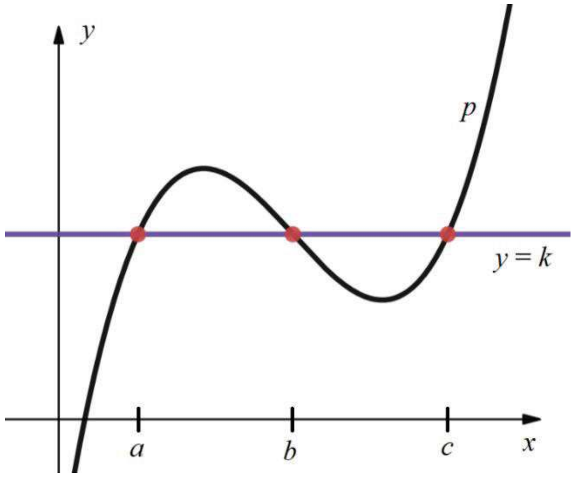
**(Solutions to *all* Q and SQ Problems appear immediately following the New Problems.)**

**SQK-A** **Proposed by Michael W. Ecker, Problem Section Editor.** Mike, age 30, and Sue, age 24, wish to marry. Unfortunately, they live in a country where the older partner may not marry the younger unless he is no more than 20% older. How many years would they have to wait to marry legally, assuming they stay in the country and the law remains in effect?

**SQK-B** **Proposed by Michael W. Ecker**. A cylindrical container is to be expanded to have 6% greater volume. If the relative increases in radius and height are to be equal, approximately what percentage increase should each get? (Hint: Differentials.)

**SQK-C** **Proposed by Michael W. Ecker.** A rectangular solid is to be constructed from fusing together 12 segments of a "pipe-cleaner" or hard "wire" for the edges. Of these, four are to be of one length $x$, four of one length $y$, and four of one length $z$. These lengths are unspecified, but the total of the lengths of the 12 wire segments cannot exceed $T$. Prove without any recourse to Calculus that the rectangular solid with the largest possible volume exists and it is the cube with side equal to $\frac{T}{12}$.

**SQK-D** **Proposed by Stephen L. Plett, Solutions Editor.** A cubic polynomial $p(x)$ has three distinct solutions to $p(x) = k$, where $k$ is a constant: $x = a$, $x = b$, $x = c$, with $a < b < c$.




If $b$ is the arithmetic mean of $a$ and $c$, prove that $(b, k)$ is the inflection point of $p(x)$.

**QK-1** **Proposed by Bryan Wilson, Contributing Editor.** An additive sequence ($a_n = a_{n-1} + a_{n-2}$) starts with positive integers $a_1$ and $a_2$. Find or describe all additive sequences that have $a_{10} = 2024$.

**QK-2** **Proposed by Albert Natian, Contributing Editor.** From a rectangular (2-dimensional) cake a regular hexagonal portion is removed. Using just one single straight cut, how can you cut the remaining cake into two equal parts? (Make reasonable assumptions about what you can do.)

**QK-3** **Proposed by Michael W. Ecker.** Solve $Tan^{-1}\left(\sin(x)\right) + x = \frac{\pi}{2}$. (Inverse trig functions.)

**QK-4** **Proposed by Michael W. Ecker**. In honor of year 2025, suppose $x$ and $y$ are nonnegative integers with $x > y$ and $x + y + xy = 2025$. How do you find the sum of all the distinct values of $x + y$?

**QK-5** **Proposed by Michael W. Ecker.** Find all rational functions $f(x) = \dfrac{ax+b}{cx+d}$ with $f^{-1} \equiv f$.

**QK-6** **Proposed by Stephen L. Plett.** Let $T_k$ represent the $k^{th}$ triangular number, and set $T_0 = 0$. Using only a bare minimum of algebra, prove by a mostly combinatorial argument that $4T_{n-1} + 6n^2 = \binom{4n}{2}$.

# New Problems

*Solutions to problems in this issue of MathAMATYC Educator are due by 25 May 2024.*

Important Invitation: *If you can solve only part(s) of a problem, send in whatever you can. An asterisk (*) indicates that the proposer did not include a solution at time of submission.*

**K-1** **Proposed By Mark Moodie, San Jacinto College, North Campus, Houston, TX.** Find the remainder when $x + x^9 + x^{25} + x^{49} + \cdots + x^{4045^2}$ is divided by $x^3 - 1$.

**K-2** **Proposed by Michael W. Ecker, inspired by QK-1 this issue.** An additive sequence $\langle a_n \rangle$ has the property that $a_n = a_{n-1} + a_{n-2}\ \forall n \geq 3$, with given initial $a_1 = a$ and $a_2 = b$.

a) Find a formula for $\sum_{i=1}^{n} a_i$ in terms only of $a$, $b$, and other known elements.

b) Show that this sum equals $a_{n+2} - 1$ (as with the Fibonacci sequence) if and only if $b = 1$.

c) Show that this sum equals $a_{n+2}$ (as with multiples of the Fibonacci sequence) if and only if $b = 0$. What is $a_{n+2}$ then in this case?

**K-3** **Proposed by Stephen L. Plett.** Choose any real number $p > 1$ and form a family of curves $F : \left\{ y_K = Kx^{\left(p^2\right)} \mid x, K\ real \right\}$. Show that a) the orthogonal trajectories constitute a family of ellipses, and b) express their common eccentricity in terms of $p$.

**K-4** **Proposed by Michael W. Ecker.** With complex variable $z = x + yi$ and complex constant $c = a + bi$, let $w = (z - c)^2$. Identify and graph all points $(x, y, w)$ for which $w$ is real-valued.

**K-5** **Proposed by Bryan Wilson.** A poorly designed clock has hour and minute hands the same length and shape. Assume the hands move continuously. There are some times during the day during which, even with precise measurement, there are two possible interpretations of the time (such as about 1:21 or 4:07). How many such times between noon and midnight will there be two possible interpretations of the time?

**K-6** **Proposed by Michael W. Ecker.** a) Given three distinct rational numbers, prove there exists an arithmetic sequence that includes them. b) Is the sequence unique? If there is more than one such arithmetic sequence, how are they related? c) Same questions for any finite number $n > 1$ of rationals.

**SPECIAL PROBLEM 2** **Proposed by Michael W. Ecker.** *For this one question alone, readers are asked to send responses directly to the Problems Editor (DrMWEcker@aol.com). There is no deadline, and responses will be gathered for at least one future issue. Note that this is not about actual calculation, but rather about strategies. Accordingly, it could mean involving students!*

Given the coordinates of four distinct points in the $x$, $y$ plane that form a quadrilateral ***QQQQ***, without graphing, devise as many ways as possible to show how one might determine whether ***QQQQ*** is convex.

# Solutions to Quicker Problems (Q) and Students' Quicker Challenges (SQ)

**(Solutions here are by the Proposer and/or Problem Section Editor, unless noted otherwise.)**

**SQK-A** The ratio of age difference to younger is currently 6/24 = 1/4 or 25% older. In $x$ years, he will still be 6 years older and her age will be 24 + $x$, so we now set the ratio to be less than or equal to 20%, or 1/5. That is, we have $\dfrac{6}{24+x} \leq \dfrac{1}{5}$. Cross-multiply to get $24 + x \geq 30$, so $x \geq 6$.

**SQK-B** We use differentials to approximate. We have $V = \pi r^2 h$ for the volume; we want $\dfrac{dV}{V} = 0.06$

to reflect 6% volume increase, with $\frac{dr}{r}=\frac{dh}{h}$. Now $\frac{dV}{V}=\frac{\pi r^2 dh+2\pi rhdr}{\pi r^2 h}=\frac{dh}{h}+2\frac{dr}{r}=3\frac{dr}{r}=0.06.$ Hence, $\frac{dr}{r}=0.02=\frac{dh}{h}$. Each of radius and height is to increase about 2%. Note: No messy decimals!

**SQK-C**

We have $x+y+z=\frac{T}{4}$, as maximal volume requires using the largest total length of wire. Suppose we hold $z$ fixed. Then $x+y=\frac{T}{4}-z$ is a constant, which we will call $2a$. So we can represent this by $x=a+\varepsilon,\ y=a-\varepsilon.$ Then the solid's volume is $V=xyz=(a+\varepsilon)(a-\varepsilon)z=(a^2-\varepsilon^2)z\le a^2z$. Hence, to obtain a maximum, we must have $y=x=a$. By the same reasoning, for any given $x$, we must also have $z=y$ for maximality. It follows that there is an absolute maximum achieved with $x=y=z$.

**SQK-D**

To begin, $p(x)=m(x-a)(x-b)(x-c)+k$, where $m$ is a constant. If $b=\frac{a+c}{2}$, then $p(x)=\frac{m}{2}(x-a)\big(2x-(a+c)\big)(x-c)$ or, after rearranging, $p(x)=\frac{m}{2}\big(2x^3-3(a+c)x^2+(a^2+4ac+c^2)x-ac(a+c)\big)+k$. Thus, we have $p'(x)=\frac{m}{2}\big(6x^2-6(a+c)x+(a^2+4ac+c^2)\big)$ and $p''(x)=m\big(6x-3(a+c)\big)$. If $p''(x)=0$, then $6x=3(a+c)$, or $x=\frac{a+c}{2}=b$. Being linear, $p''(x)$ changes sign at $x=b$. Therefore, the concavity of the graph of $y=p(x)$ does indeed change at $(b,k)$.

**QK-1**

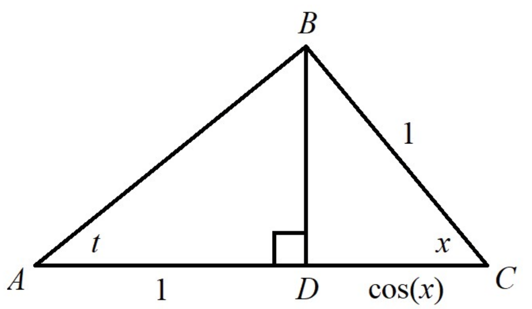
**Solution by the Problems Editor.** We have $a_n=a_{n-1}+a_{n-2}\ \forall n\ge 3$, with given initial $a_1=a$ and $a_2=b$. The Fibonacci sequence $\langle F_n\rangle$ is the special case with $a=b=1$. Every such additive sequence $<a_n>$ with $a_1=a$ and $a_2=b$ satisfies $a_n=F_{n-2}a+F_{n-1}b$. (Just calculate a few values and the pattern will be clear.) Since $F_8=21,\ F_9=34$ we are solving $21a+34b=2024$. From $0\le b\le\left\lfloor\frac{2024}{34}\right\rfloor=59$, a simple computer program reveals precisely two solutions for $(a,b)$: (64, 20) and (30, 41).

**QK-2**

The cut is the straight line joining the centers of rectangle and hexagon. As both regular hexagon and rectangle are centrally symmetric, the straight line through their centers bisects the regions. (Bryan Wilson points out that this assumes hexagon's center is not at the center of the rectangle – a trivial case.)

**QK-3**

**Solution I (More Geometric) by Proposer.** With $f(x)=Tan^{-1}(\sin(x))+x$, we then have $f'(x)=\frac{\cos(x)}{\sin^2(x)+1}+1>0\ \forall x\in\mathbb{R}$. Therefore, $f$ is increasing, so our equation can have at most one solution. If we set $t=Tan^{-1}(\sin(x))$, then $\tan(t)=\sin(x)$, leading to the diagram at the right, with $BD=\tan(t)=\sin(x)$. As a consequence, we also have $AB=\sqrt{1+\sin^2(x)}$.




Now, $Tan^{-1}(\sin(x))+x=\frac{\pi}{2}$ implies for the right $\Delta ABC$ that $\left(\sqrt{1+\sin^2(x)}\right)^2+1^2=(1+\cos(x))^2$. Simplifying leads to $\cos^2(x)+2\cos(x)=1+\sin^2(x)=2-\cos^2(x)$, or $\cos^2(x)+\cos(x)-1=0$. Hence, $\cos(x)=\frac{-1+\sqrt5}{2}=s$, and so $x=Cos^{-1}(s)$ (where $s=\frac{-1+\sqrt5}{2}$ relates to the golden mean).

**Solution II (Quicker) by Albert Natian.** From the identity $Tan^{-1}(y)+Cot^{-1}(y)=\frac{\pi}{2}$, it follows that $x=Cot^{-1}(\sin(x))$, so $\cot(x)=\sin(x)$. Change $\cot(x)$ to $\frac{\cos(x)}{\sin(x)}$ and get $\cos(x)=\sin^2(x)$, or $\cos(x)=1-\cos^2(x)$, resulting in the same $\cos^2(x)+\cos(x)-1=0$, yielding $x=Cos^{-1}(s)$ just as above.

**QK-4**

Add 1 to get $x+y+xy+1=2026$. Now factor each side to get $(x+1)(y+1)=2\cdot 1013$. The number of divisors of 2026 is therefore $2\cdot 2=4$, and these divisors are 1, 2, 1013, 2026. They pair together as $2026\cdot 1$ and $1013\cdot 2$. Each of these is of the form $(x+1)(y+1)$. Associated with each pair $(x+1,y+1)$ is the original $(x,y)$, so there are just two pairs to make up the $x+y$ to sum, namely $2025+0$ and $1012+1$. The sum of these is 3,038.

**QK-5** With $y=\dfrac{ax+b}{cx+d}$ we solve for $x$: $(cx+d)y=ax+b$, so $(cy-a)x=-dy+b$, giving $x=\dfrac{-dy+b}{cy-a}$. Hence, after swapping $x$ and $y$, re-labeling, and equating, we get $f^{-1}(x)=\dfrac{-dx+b}{cx-a}=\dfrac{ax+b}{cx+d}$. It would be tempting to confidently declare that the required condition is that $d=-a$, but that would be incomplete, as the self-invertible example of $f(x)=x$ shows ($d=a=1$). Now, if $c\neq 0$, then we do conclude that $d=-a$. However, if $c=0$, then write $f(x)=Ax+B$ and so $f^{-1}(x)=\dfrac{x-B}{A}=Ax+B$. That is, $\dfrac{1}{A}=A$ *and* $\dfrac{-B}{A}=B$. From the first, $A=\pm 1$, *and* from the second, $B(A+1)=0$, giving $B=0$ *or* $A=-1$. There are four sub-cases: a) $A=1, B=0$, implying $f(x)=x$; b) $A=1, A=-1$, a contradiction; c) $A=-1, B=0$, implying $f(x)=-x$; d) $A=-1, A=-1$, implying $f(x)=-x+B$. We can assemble the results to conclude that $f(x)=\dfrac{ax+b}{cx+d}$ is self-invertible if and only if $d=-a$, or $f(x)=x$, or $f(x)=-x+B$ (for all $B$, including 0). Note that the known example of $f(x)=\dfrac{b}{x}$ satisfies $d=-a=0$.

**QK-6** We rewrite the equation as $\binom{4}{1}\left(\frac{n(n-1)}{2}\right)+\binom{4}{2}(n\cdot n)=\binom{4n}{2}$. Then also $\binom{4}{1}\binom{n}{2}+\binom{4}{2}\binom{n}{1}\binom{n}{1}=\binom{4n}{2}$. Now consider a set $S$ which is the union of four disjoint subsets, which we call subcategories, each containing $n$ elements. The right side counts the total number of ways to select two elements from $S$. The left side does the same, depending on whether we choose both elements from the same subcategory, or select one element from each of two of the subcategories.

# Solutions to Problems from Previous Issues

<u>Solver List Update</u>: *Owing to an oversight or a submission coming close to deadline, Mark Moodie was not credited for some of his contributions last issue. In particular, he submitted correct solutions to problems G-2 and G-5.*

## Ant Trails

**G-6 Proposed by Albert Natian.** An ant is initially at vertex $A$ of a cube. Whenever it finds itself at a vertex of the cube, it chooses, with equal probability, one of the three edges emanating from said vertex. It then moves along that edge in one second to move to the edge's other vertex. On average, how long will it take the ant for its first return to vertex $A$?

**Solution by Troy Williamson, Texas State Technical College, Abilene, TX.** Let vertex $A$ be placed at the origin: $A=(0,0,0)$, and let the cube measure 1 unit in all directions, such that the diagonally opposite corner of the cube lies at $(1,1,1)$. The task is to determine the expected number of moves, thus the number of seconds that will elapse, before the ant returns to $A$. The eight vertices of the cube can be separated into four disjoint sets. The first set contains those vertices from which no further movement is needed: $S_0=\{A\}$. The second set contains those vertices from which the ant could arrive at $A$ in one move: $S_1=\{(1,0,0),(0,1,0),(0,0,1)\}$. The third set contains those vertices from which the ant could arrive at $A$ in two moves: $S_2=\{(1,1,0),(1,0,1),(0,1,1)\}$. The last set contains the remaining vertex, from which the ant would need three moves to arrive at $A$: $S_3=\{(1,1,1)\}$. Let $N_i$ represent the expected number of moves from a vertex which is a member of $S_i$. We can see that $N_0=0$, since the ant has arrived at its destination. (This does not count the initial placement of the ant, of course.) From a vertex in $S_1$, there is a probability of $\frac{1}{3}$ that the ant chooses to move to $A$, and there is a probability of $\frac{2}{3}$ that the ant moves to a vertex that is a member of $S_2$. Thus, $N_1=\frac{1}{3}(1)+\frac{2}{3}(1+N_2)$. From a vertex in $S_2$, there is a probability of $\frac{2}{3}$ that the ant will move to a vertex which is an element of $S_1$, and there is a probability of $\frac{1}{3}$ that the ant will move to the vertex that is the sole element of $S_3$. Thus, $N_2=\frac{2}{3}(1+N_1)+\frac{1}{3}(1+N_3)$ = $1+\frac{2}{3}N_1+\frac{1}{3}N_3$. Finally, from $(1,1,1)\in S_3$, the ant has no choice but to move to a member of $S_2$, so $N_3=1+N_2$. Solving this system of three equations in three unknowns tells us $N_1=7$, $N_2=9$, and $N_3=10$. Because the ant begins at $A$, on the first move it has no choice but to move to a member of $S_1$. So, the expected number of moves from $A$ is $1+N_1=1+7$ = 8, and as the ant makes each move in 1 second, the average time elapsed before the ant first returns to $A$ is 8 seconds.

**Also solved by the proposer.**

## More Ant Trails

**H-1 Proposed by Michael W. Ecker.** An ant can be at either $x = 0$ or $x = 1$ on the $x$-axis. It starts at $x = 0$. A coin is flipped repeatedly. If heads, it moves right to $x = 1$, if possible. If tails, it moves left to $x = 0$, if possible. (If not possible, stay put in either case.) What is the average number of coin flips needed to move from 0 to 1 and back to 0?

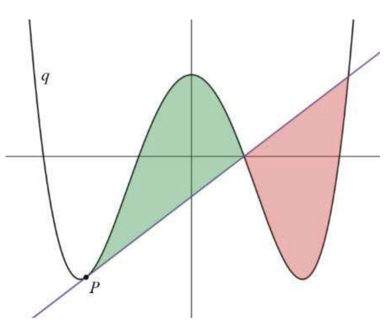
**Solution I by Austin Jones, Wake Technical Community College, Raleigh, NC; with essentially similar solutions by Michael C. Faleski, Delta College, University Center, MI; Raymond N. Greenwell, Hofstra University, Hempstead, NY; Troy Williamson; and the proposer.** A given successful there-and-back-again consists of some number of tails, then one head, then some number of heads, and finally a single tail; i.e., $TT\cdots T\mathbf{H}H\cdots H\mathbf{T}$, where bold means the ant moved. The briefest success is one head followed by one tail (***HT***), giving 2 flips as the minimum number of flips. If there are $k$ total flips in a given success, then there are $k-1$ possible locations for the first head, ***H*** (anywhere except last, which must be ***T***), which fixes all other flips. Thus, the probability of exactly $k$ flips is $P(k) = (k-1)\left(\frac{1}{2}\right)^k$. Therefore, the expected/average number of flips is $E = \sum_{k=2}^{\infty} k \cdot P(k) = \sum_{k=2}^{\infty} k \cdot (k-1)\left(\frac{1}{2}\right)^k = \frac{1}{4} \cdot \sum_{k=2}^{\infty} k \cdot (k-1)\left(\frac{1}{2}\right)^{k-2}$.

Now consider $f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $0 < x < 1$, and notice $f''(x) = \frac{2}{(1-x)^3} = \sum_{k=2}^{\infty} k(k-1)x^{k-2}$. Hence $E = \frac{1}{4} f''\left(\frac{1}{2}\right) = \frac{1}{4} \cdot \frac{2}{(1-\frac{1}{2})^3} = 4$.

**Solution II by Albert Natian.** The average time going from 0 to 1 to 0 is the sum of the average time going from 0 to 1 and the average time going from 1 to 0. Because the probability of going from 0 to 1 (which is the same as going from 1 to 0) is 1/2, then the average time going from 0 to 1 is 2, and so is the average time going from 1 to 0. So, the answer is 2+2=4.

## Quartic Quandaries

**H-2 Proposed by Stephen L. Plett.** An even, monic, quartic polynomial function $q$ has four distinct real zeroes.




(a) Find every third-quadrant point $P$ on $q$'s graph where the line tangent to the curve at $P$ passes through an $x$-intercept of the curve. (b) Find the ratio of the positive zeroes which makes the regions bounded by the tangent line and the curve have equal area.

**Composite of solutions to Part (a) by Raymond N. Greenwell; Austin Jones; and the proposer.** An even, monic quartic polynomial function $q$ with four distinct real zeroes has the form $q(x) = (x^2 - a^2)(x^2 - b^2) = x^4 - (a^2 + b^2)x^2 + a^2b^2$, where $\pm a$ and $\pm b$ are the zeros. Denote a third-quadrant point $P$ on $q$'s graph where the line tangent to the curve at $P$ passes through an $x$-intercept of the curve by $(r, s)$. The tangent line there has equation $y - s = q'(r) \cdot (x - r)$, with $q'(r) = 4r^3 - 2(a^2 + b^2)r$ and $s = q(r) = (r^2 - a^2)(r^2 - b^2)$. In order for this tangent line to pass through $(a, 0)$, the reader should confirm that we eventually get $(r-a)^2\left(3r^2 + 2ar - b^2\right) = 0$.

(<u>Problems Editor's Note</u>: I got this exact same result, from scratch, using a mildly different approach. Steve and I ask you to check all the algebra for yourself.) Now the quadratic formula says $r = \frac{-a \pm \sqrt{a^2+3b^2}}{3}$, and we must choose the negative sign in order to have point $P$ in quadrant III. Finally $s = q(r) =$

$$\frac{-1}{81}\left(10a^4 + 10a^3\sqrt{a^2+3b^2} - 60a^2b^2 + 6ab^2\sqrt{a^2+3b^2} + 18b^4\right)$$

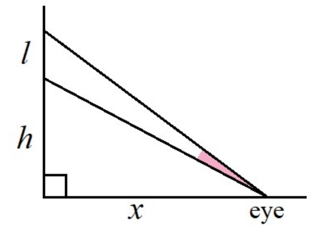
Similar calculations can be used for the tangent lines passing through $(-b, 0)$ and $(b, 0)$.

**No solutions for Part (b) were submitted by others.** (<u>Problems Editor's Note</u>: Actually, I found an archived solution from the proposer showing the claimed result that $s = r\sqrt{8} = 2\sqrt{2} \cdot r$. If anybody cares to write me at DrMWEcker@aol.com I will send you a copy of the Word document from Steve.)

## Get The Picture?

**H-3 Proposed by Michael W. Ecker.** On a museum wall (perpendicular to ground), a painting has length down equal to $l$, with its bottom $h$ high above eye level. You choose how far $x$ to stand from the wall. Your only concern is to get the best view, by having the angle that the painting subtends on the eye be as large as possible. How far from the wall should you stand?

**Solution I by Ivan Retamoso, Borough of Manhattan Community College, New York, NY; and (independently) Mark Moodie, San Jacinto College, North Campus, Houston, TX; with essentially similar solutions by Michael C. Faleski; Raymond N. Greenwell; and Austin Jones.** Start by labeling the angles with α and θ as shown in the figure below, noticing $0<\theta<\frac{\pi}{2}$.




The goal is to get the best view, which will happen when $\theta$ is as big as possible. If $x \to 0$, then $\theta \to 0$ (observer very close to the wall). If $x \to \infty$, then $\theta \to 0$ (observer very far from the wall). Note that $\tan\theta$ is well known to be strictly increasing for $0<\theta<\frac{\pi}{2}$, easily confirmed by its derivative, $\sec^2\theta$ being positive for $0<\theta<\frac{\pi}{2}$. Hence, finding the value of $x$ that will maximize $\theta$ is equivalent to maximizing $\tan\theta$. From the diagram, $\tan(\alpha+\theta)=\frac{l+h}{x}$ so $\frac{\tan\alpha+\tan\theta}{1-\tan\alpha\cdot\tan\theta}=\frac{l+h}{x}$ or $\frac{\frac{h}{x}+\tan\theta}{1-\frac{h}{x}\cdot\tan\theta}=\frac{l+h}{x}$. Cross-multiply, multiply through by $x$, and then isolate $\tan\theta$ to produce $\tan\theta=\frac{xl}{x^2+hl+h^2}$. Set its derivative with respect to $x$ equal to zero to find the critical number for $x$, which will determine the unique maximum: $\frac{l(x^2+hl+h^2)-2x(xl)}{\left(x^2+hl+h^2\right)^2}=0$. Since the denominator is positive, $lx^2+hl^2+lh^2-2x^2l=0$ or $x^2=hl+h^2=h(l+h)$, which gives the answer $x=\sqrt{h(l+h)}$.

**Solution II by the proposer.** Set $\beta=\alpha+\theta$ as the larger acute angle near the eye so that the subtended angle $\Theta$ is the difference of two angles in the two right triangles. We then have $\theta=\beta-\alpha=Cot^{-1}\left(\frac{x}{h+l}\right)-Cot^{-1}\left(\frac{x}{h}\right)$. (Note: I have used the inverse cotangent instead of the inverse tangent to avoid the variable of interest being in the denominators.) The situation clearly has $\Theta$ small for $x$ getting very small or very large, so we expect a unique value of $x$ with maximal angle precisely where $\frac{dx}{d\theta}=0$, with no further testing needed. Set $\frac{dx}{d\theta}=-\frac{h+l}{(h+l)^2+x^2}+\frac{h}{h^2+x^2}=0$. Solve for $x$ by cross-multiplying, gathering like terms, and common factoring to get: $x^2=h(h+l)$ so $x=\sqrt{h(h+l)}$.

Problems Editor's Notes: 1. We discarded the negative $x$-value in both solutions, but we could have kept it by treating it formally by thinking of the other side of the wall as a kind of mirror room. 2. On a more significant level, just before our deadline, Contributing Editor Albert Natian reported that there is a non-Calculus, purely geometric proof to show $x=\sqrt{h(h+l)}$. In fact, I briefly saw that proof decades ago, but did not absorb it then. To quote Albert: "The solution is precisely the point $(x, 0)$ at which a circle passing through the points $(0, h)$ and $(0, h+l)$ is tangent to the $x$-axis." Since then, I've studied his proof personally and confirmed the accuracy of the information and the underlying geometry theorem. Unfortunately, this came in a bit too close to deadline. I will hold this information for possible inclusion in a future column as a **Solution III**, which I probably would have dubbed as our preferred solution.

## A Triggy Inequality

**H-4 Proposed by Michael W. Ecker.** Which is larger, $\cos(\sin(x))$ or $\sin(\cos(x))$? (Prove it.)

**Solution by Austin Jones.** Let $f(x)=\cos(\sin(x))$ and $g(x)=(\sin(\cos x))$. Note that $f(0)=1 > g(0)=\cos(1)$. The period of $f$ and $g$ is $2\pi$, and both are even: $f(-x)=\cos(\sin(-x)) = \cos(-\sin(x)) = \cos(\sin(x)) = f(x)$, and $g(-x)=\sin(\cos(-x)) = \sin(\cos(x)) = g(x)$. Hence it suffices to show $f(x)>g(x)$ on $(0,\pi]$. But on $\left[\frac{\pi}{2},\pi\right]$, $f(x)>0$ while $g(x)<0$, so it suffices to show that $f(x)>g(x)$ on $\left(0,\frac{\pi}{2}\right)$. Assume that $0<x<\frac{\pi}{2}$ and recall $x>\sin x$ for $x>0$. Since $\cos(x)>0$, then $\cos(x) > \sin(\cos x))=g(x)$, and notice that $\frac{d}{dx}(\cos(x))=-\sin(x) < 0$. Since $\sin(x)< x <\frac{\pi}{2}$, then $f(x)=\cos(\sin(x)) > \cos(x)$, and transitivity establishes the result.

**Also solved (many similarly) by Jackson Dahlin, Sterling Davis, and Jack Rainey (students), Taylor University, Upland, Indiana; Ivan Retamoso; Troy Williamson; and the proposer.**

## From Recursion to Function

**H-5 Proposed by Albert Natian.** a) Find the $n^{\text{th}}$ term of the sequence $\langle a_n\rangle$ with $a_1=a_2=1$ and, $\forall n\geq 3$, $a_n=\frac{a_{n-1}a_{n-2}}{\sqrt{a_{n-1}{}^2+a_{n-2}{}^2}}$. b) Same question: $\forall n\geq 3$, $a_n=a_{n-2}\left(1+\frac{a_{n-3}}{a_{n-1}}\right)$, $a_0=a_1=a_2=1$.

**Solution to Part (a) by Raymond N. Greenwell, with similar solutions by Austin Jones and the proposer.** We note that $a_1=a_2=1$, and $a_n=\frac{a_{n-1}a_{n-2}}{\sqrt{(a_{n-1})^2+(a_{n-2})^2}}$. Thus $a_3=1\cdot\frac{1}{\sqrt{1^2+1^2}}=\frac{1}{\sqrt{2}}$. We will prove by induction that

$a_n = \frac{1}{\sqrt{F_n}}$ where $F_n$ is the $n^{\text{th}}$ Fibonacci number ($F_1 = F_2 = 1$; $F_n = F_{n-1} + F_{n-2}$, $n \geq 3$). We have already shown this to be true for $n = 1$, 2, and 3. Suppose we assume it to be true for $n = 1, 2, \ldots, k$ for some positive integer $k$. Then $a_{k+1} = \frac{a_k\, a_{k-1}}{\sqrt{(a_k)^2 + (a_{k-1})^2}} = \frac{\frac{1}{\sqrt{F_k}} \cdot \frac{1}{\sqrt{F_{k-1}}}}{\sqrt{\frac{1}{F_k} + \frac{1}{F_{k-1}}}} = \frac{\frac{1}{\sqrt{F_k\, F_{k-1}}}}{\sqrt{\frac{F_{k-1}+F_k}{F_k\, F_{k-1}}}} = \frac{1}{\sqrt{F_{k+1}}}$, and the assertion is true.

**Solution to Part (b) by Austin Jones; with similar solutions by Raymond N. Greenwell and the proposer.** Getting a common denominator and expanding the product, we have: $a_n = a_{n-2}\left(1 + \frac{a_{n-3}}{a_{n-1}}\right) = a_{n-2}\left(\frac{a_{n-1} + a_{n-3}}{a_{n-1}}\right) = \frac{a_{n-1}\, a_{n-2} + a_{n-2}\, a_{n-3}}{a_{n-1}}$. So $a_n\, a_{n-1} = a_{n-1}\, a_{n-2} + a_{n-2}\, a_{n-3}$. Define the sequence $b_1 = a_n\, a_{n-1}$. Then $b_n = b_{n-1} + b_{n-2}$ and begins $b_1 = (1)(1) = 1$, $b_2 = (1)(1) = 1$, so $b_n$ satisfies the Fibonacci recurrence. Thus, $a_n = \frac{a_{n-1}\, a_{n-2} + a_{n-2}\, a_{n-3}}{a_{n-1}} = \frac{F_{n-1}}{a_{n-1}} \quad \frac{F_n\, a_{n-1}}{F_{n-1}} = \frac{F_n\, F_{n-2}}{F_{n-1}\, a_{n-2}} = \ldots = \frac{F_n\, F_{n-2} \cdots}{F_{n-1}\, F_{n-3} \cdots}$ where the products end at either $F_2$ or $F_1$ depending on whether $n$ is even or odd. Concisely, but in no way more illuminating we have:

$$a = \begin{cases} \prod_{j=1}^{n/2} \frac{F_{2j}}{F_{2j-1}}, & n \text{ even} \\ \prod_{j=1}^{(n-1)/2} \frac{F_{2j-1}}{F_{2j}}, & n \text{ odd} \end{cases}.$$

## Digits Displayed

**H-6 Borrowed by the Problems Editor**. Let $x$ be any positive integer. a) Prove that for any digit $d$, there exists an integer multiple of $x$ that contains $d$ in its decimal representation. b) Can you prove an even stronger result?

**Solution by Michael W. Ecker.** We prove b), a much stronger result that includes part a). That is, given any positive integer $x$, there exists a positive integer $y$ such that the product $xy$ contains *all* ten digits. Let $B_1 = 9876543210$ in positional notation, $B_2 = 98765432109876543210$, and so on, using concatenation each time, so that generally, $B_k = k$ copies of $B_1$, concatenated. Now, consider the $x$ numbers $B_1$ through $B_x$. If one of them is a multiple of $x$, we are done (and we call that multiple $y$). Otherwise, we have $x$ numbers, each with a nonzero residue (mod $x$). That is, each one of the $x$ numbers produces a nonzero remainder when divided by $x$. But, there are at most $x-1$ distinct nonzero remainders. By the pigeon-hole principle, at least two of the $x$ numbers have the same nonzero residue $r$ (mod $x$), say $B_k$ and $B_j$, with $j < k$. That is, $B_k = ax + r$ and $B_j = bx + r$. Hence, $B_k - B_j = (ax+r) - (bx+r) = (a-b) \cdot x$ is a multiple of $x$, but also $B_k - B_j = 10^{10j} B_{k-j}$ has every digit.

**Part a) was also solved by Troy Williamson.**

**Michael W. Ecker** had a 45-year career as a mathematics professor, retiring from Penn State University's Wilkes-Barre campus in 2016. With a PhD in mathematics from the City University of New York (1978), he has been published 500+ times as a mathematician or computer journalist. Mike also served on the national committees responsible for creating the MAA's competitive national exams, and was the Founding Problem Section Editor of *The AMATYC Review* (1981-1997). As a recreational mathematician, he published his own newsletter, *Recreational & Educational Computing* (**REC**, 1986-2007). For free PDF copies of **REC**, visit https://dr-michael-ecker.weebly.com.

**Stephen L. Plett** retired in 2020 after 40 years as a mathematics professor, mostly at Fullerton College. He holds two master's degrees: applied mathematics from the University of California, Riverside (1980) and mathematics for collegiate teaching from the California State University, Fullerton (1988). Steve has authored introductory textbooks in Differential Equations and Linear Algebra, a few journal articles, and has contributed to publications in problem solving for 35 years, including being Solutions Editor of *The AMATYC Review* for Mike and his successor as Problem Section Editor (1997-2008).

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