Heuristic Method for Minimizing Distance without using Calculus and Its Significance

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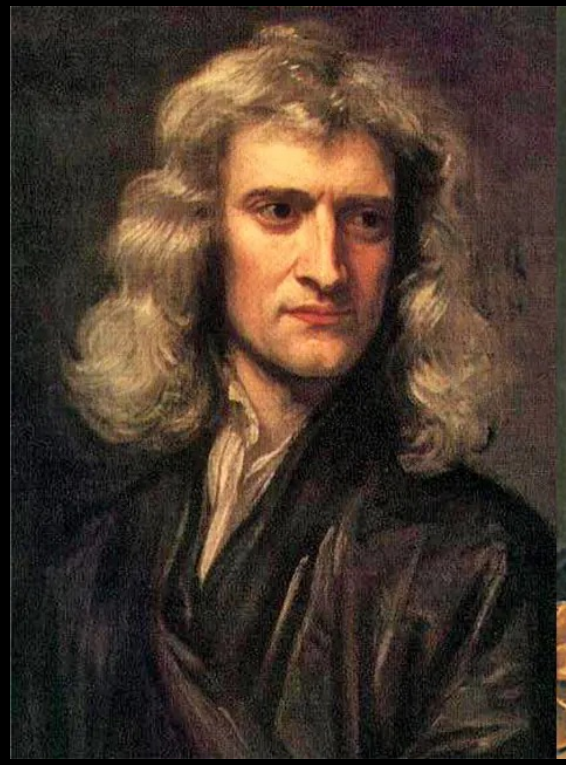
Since our intention is not to ignore Calculus as a fundamental tool for solving applied optimization problems, let's start by solving a classical basic problem about minimizing a distance using Calculus, then we will solve the same problem using our method, and gradually we will cover more challenging cases.



Isaac Newton

Gottfried Leibniz

Problem 1

Paul's house is located at point A, the farm of his grandmother is located at point B, there is a river as shown in the figure below, every morning, Paul needs to go to the river, get water and bring it to his grandmother's farm, what is the length of the shortest path Paul should follow? See figure 1 below.

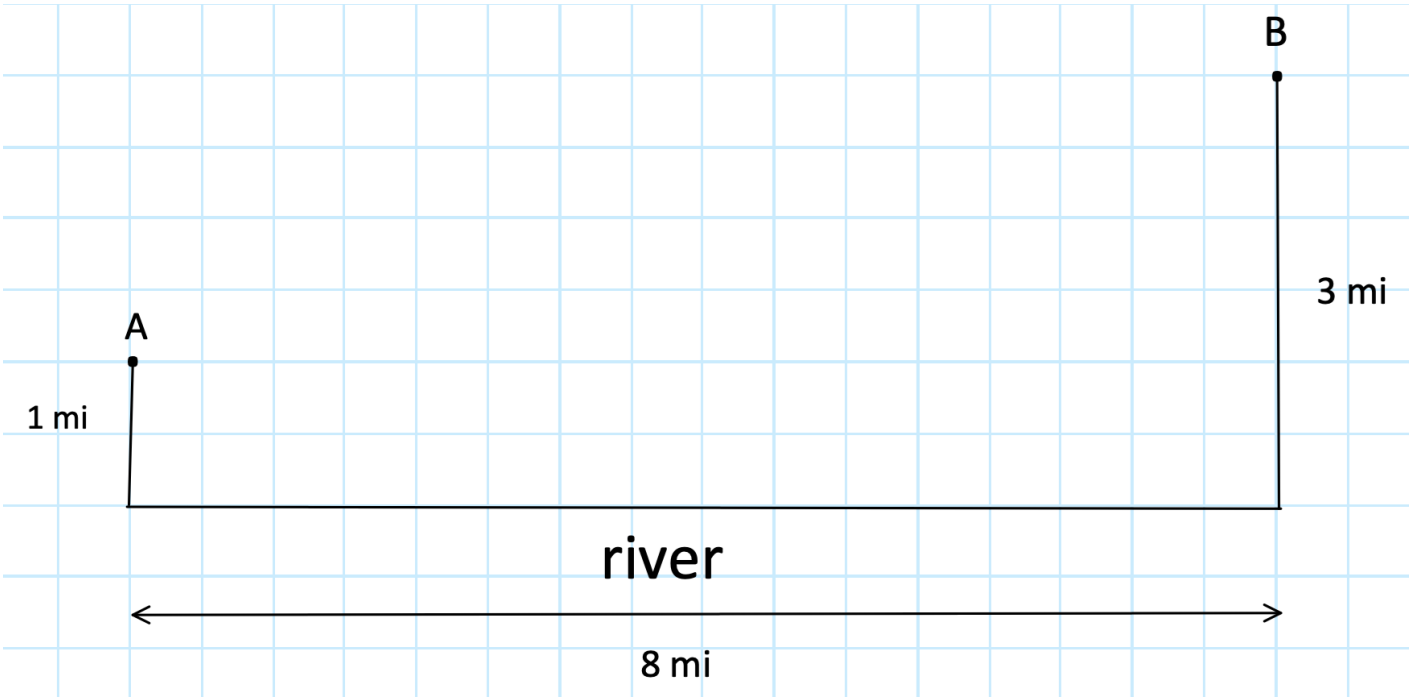


Figure 1: Classical problem in Calculus.

A common solution to this classical problem using Calculus goes like this, let C be the point where Paul will reach the river, let x be the distance between O and C, see figure 2 below.

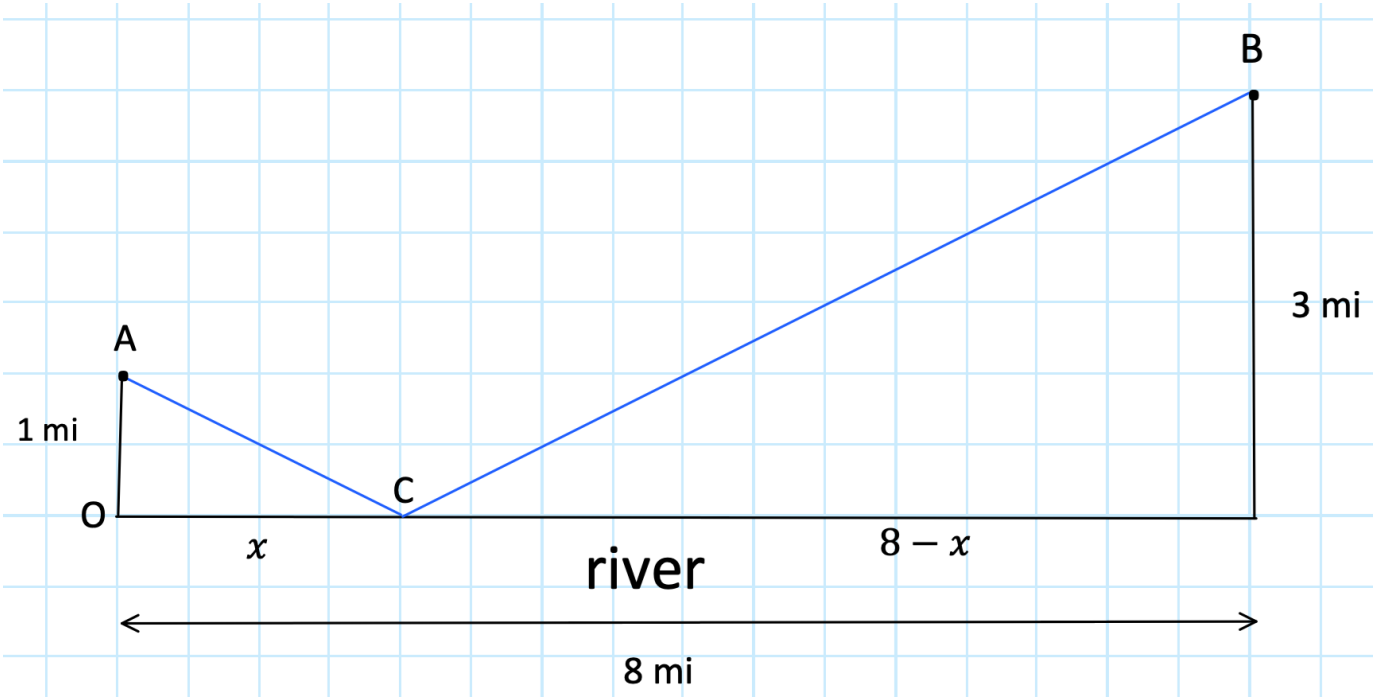


Figure 2: Common solution using Calculus.

then we must minimize the distance $AC + CB$, equivalently, we must minimize the function:

$$f(x) = \sqrt{1+x^2} + \sqrt{(8-x)^2+9} \tag{1}$$

Taking the Derivative of $f(x)$, setting it equal to zero, and solving for x we obtain:

$$\frac{1}{2\sqrt{1+x^2}}\,2x + \frac{1}{2\sqrt{(8-x)^2+9}}\,2(8-x)(-1) = 0 \tag{2}$$

$$\frac{x}{\sqrt{1+x^2}} = \frac{8-x}{\sqrt{(8-x)^2+9}} \tag{3}$$

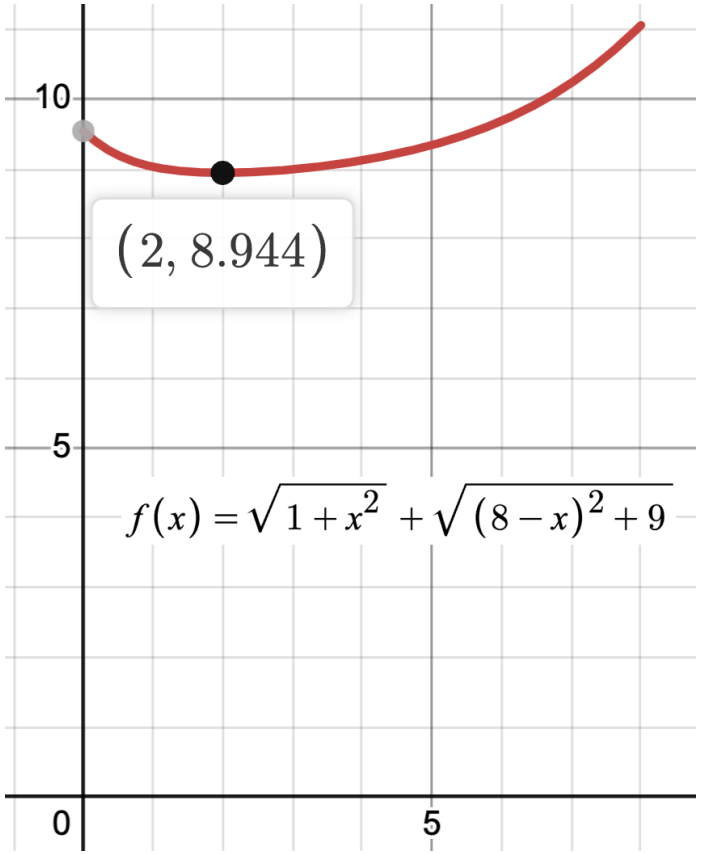
$$\frac{x^2}{1+x^2} = \frac{(8-x)^2}{(8-x)^2+9} \tag{4}$$

$$x^2(8-x)^2+9x^2 = (8-x)^2+x^2(8-x)^2 \tag{5}$$

$$9x^2 = (8-x)^2 \tag{6}$$

$$3x = 8-x \tag{7}$$

$$x = 2 \tag{8}$$



How can we solve this problem differently?

Let's use "The Reflection Principle"

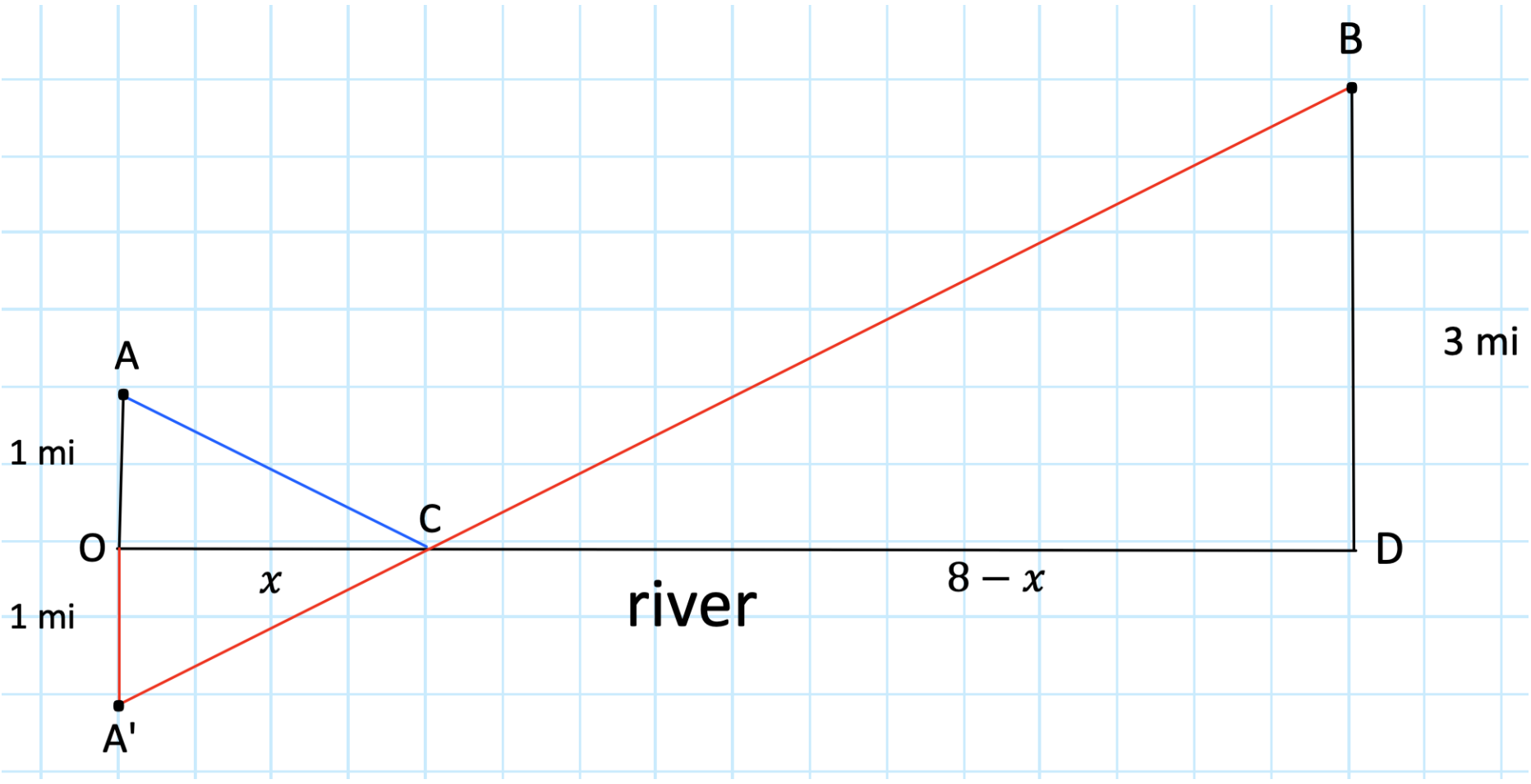


Figure 3: First application of The Reflection Principle.

$$\frac{1}{x} = \frac{3}{8 - x} \tag{12}$$

$$3x = 8 - x \tag{13}$$

$$x = 2 \tag{14}$$

Which leads to the shortest path $A \rightarrow C \rightarrow B$ having as its length:

$$\sqrt{1 + 2^2} + \sqrt{(8 - 2)^2 + 9} = 8.94\ miles. \tag{15}$$

Thanks to the Reflection Principle we solved problem 1!

The Reflection Principle

The minimum distance between two points is the length of a straight path that connects them, if we want to go from a point A to a point B via a sequence of straight paths touching once a line L, then ideally we may start at point A' the reflection of point A about the line L, this assumption can be made because moving towards line L from A is equivalent, in terms of distance covered, to move towards line L from A' given that the triangles formed above and below L are always congruent, this in turn leads to the equality of the reference angles formed by the paths and the line L, which makes the connection with the principle of reflection in physics, which shows that if a beam of light is aimed at a mirror positioned at line L, it will generate equal angles of incidence an reflection.

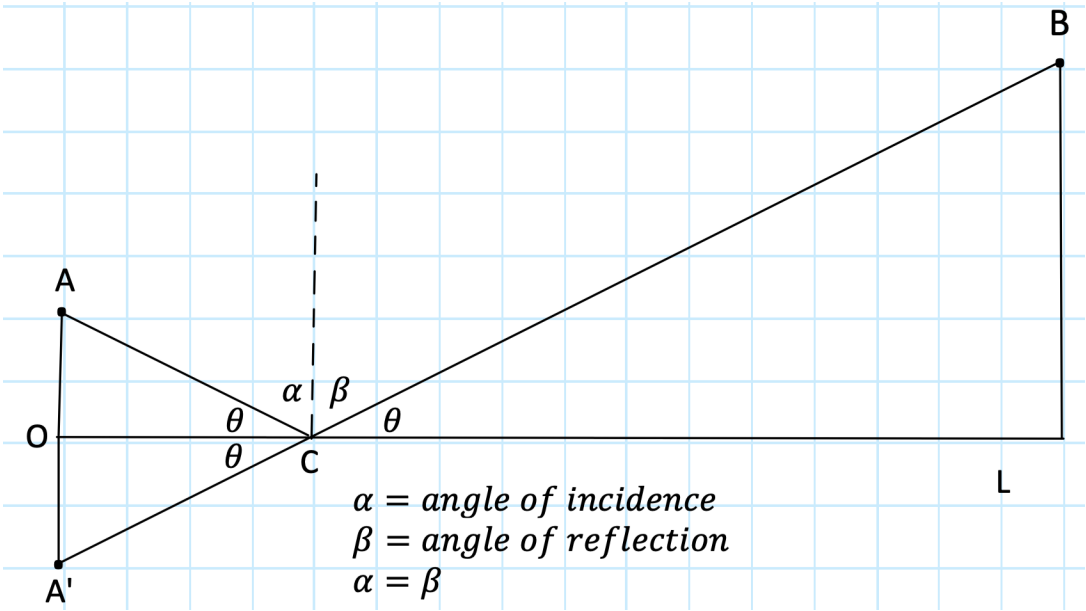


Figure 4: Explanation of The Reflection Principle.

Our alternative solution using "The Reflection Principle" has the following Educational and Computational advantages:

- It is Geometrically constructive; one can draw the shortest path using only straight edge and compass, this is particularly important because as Calculus instructors we can ask our students to build the "shortest path" as an in-class activity.

- It is algebraically simple, basically, we need to solve a rational equation which reduces itself to a linear equation, this means that this exercise could be given to students in a Precalculus class as a special group project.

- It shows that the reference angles formed by the path components with the horizontal line that represents the river are the same.

- It can be extended via "The Reflection Principle" to the solution of more challenging scenarios as we will show later.

Let's consider now a more challenging scenario where in order to go from point A to point B it is needed to touch once two perpendicular lines.

Problem 2

Suppose you have points A(1,6) and B(5,2) on an xy coordinate system, find the length of the shortest path to go from A to B touching first y-axis once and then x-axis once, see figure 6 below.

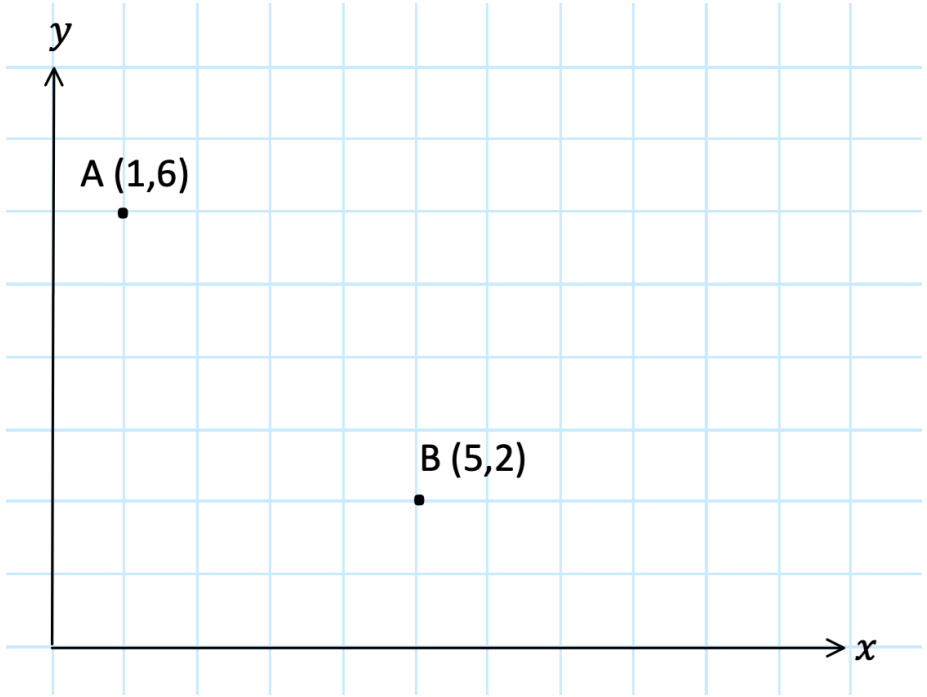


Figure 5: First extension of our basic problem.

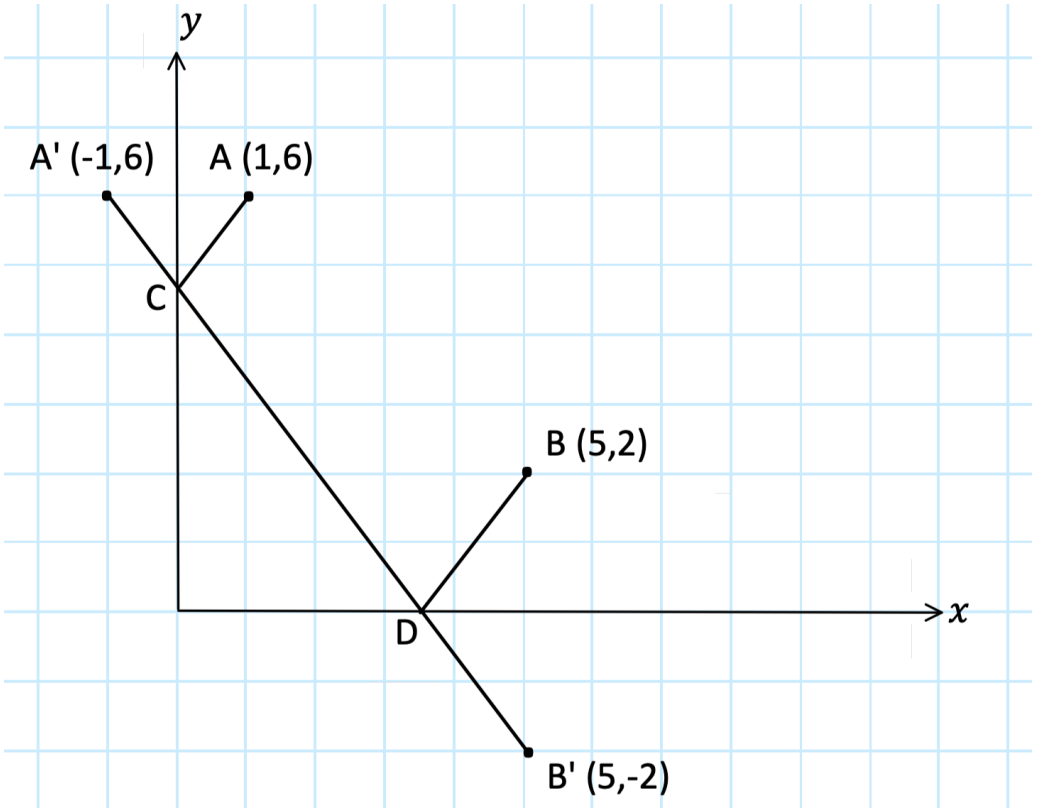
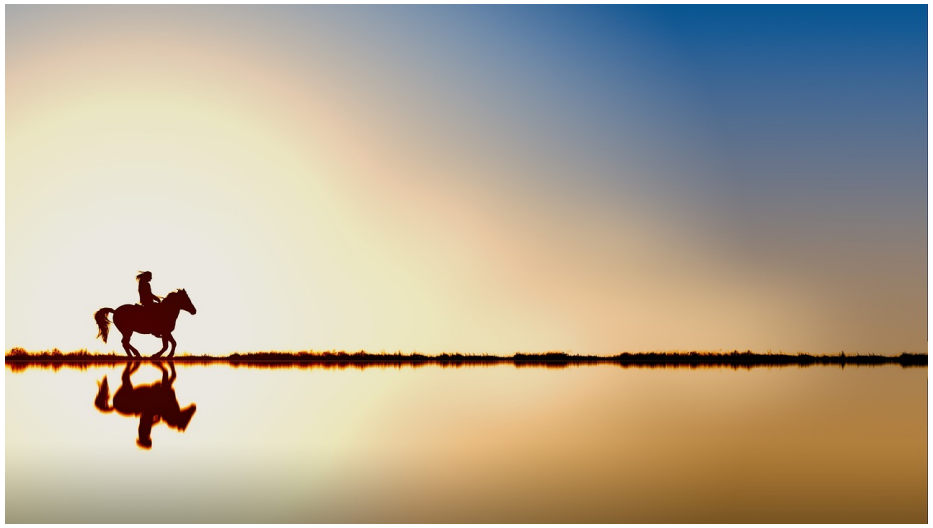


Figure 6: Second application of The Reflection Principle.

The equation of the line passing through the points $A'(-1,6)$ and $B'(5,-2)$ is $y = -\frac{4}{3}x + \frac{14}{3}$ with y-intercept $C = \left(0, \frac{14}{3}\right)$ and x-intercept $D = \left(\frac{7}{2}, 0\right)$ this determines the shortest path, then the length of the shortest path $A \to C \to D \to B$ can be calculated using the distance formula for the points A' and B' as shown below:

$$\sqrt{(5-(-1))^2 + (-2-6)^2} = 10\ units.$$



Thanks to the Reflection Principle we solved problem 2!

Remarks

- Solving Problem 2 using Calculus, may in general, require the minimization of a distance objective function that after being differentiated, could lead to a complicated algebraic equation to solve involving radicals, since we would have to deal with the sum of distance formulas as shown in the first solution of Problem 1.
- Our solution of problem 2 only used knowledge about equations of lines in an xy coordinate system and location of x-intercepts and y-intercepts, which makes problem 2 suitable for a group project once "The Reflection Principle" is explained.

How our solutions can be associated to real-life events

After solving problem 1 and problem 2 using "The Reflection Principle" a pattern emerges and a natural question comes to our minds, are these "shortest paths" related to any events in real life? and the answer to this question is yes, when we consider the path of light as an electromagnetic wave traveling in our universe, it turns out since light travels minimizing time at a constant speed then if we consider the pairwise perpendicular lines in problems 1 and 2 as mirrors, a beam of light from a flash light using the angles we found in the solutions of problems 1 and 2 will follow the same paths as we found for our solutions in problems 1 and 2, this can be confirm experimentally

Additionally, if we think of the game of billiards If we hit a billiard ball with the stick, using angles found in our solutions to problems 1 and 2, the paths followed by the billiard ball before and after hitting the edges of the billiard table will be the same as the “shortest paths” that we found in our solutions to problems 1 and 2, this of course, assuming a frictionless billiard table.

There is a way to link how light travels a as wave and the way a billiard ball hits and bounces off a crease:

A ball hitting a crease and bouncing off, acts as a wave of light reflecting off a mirror as seen in figure 8 below:

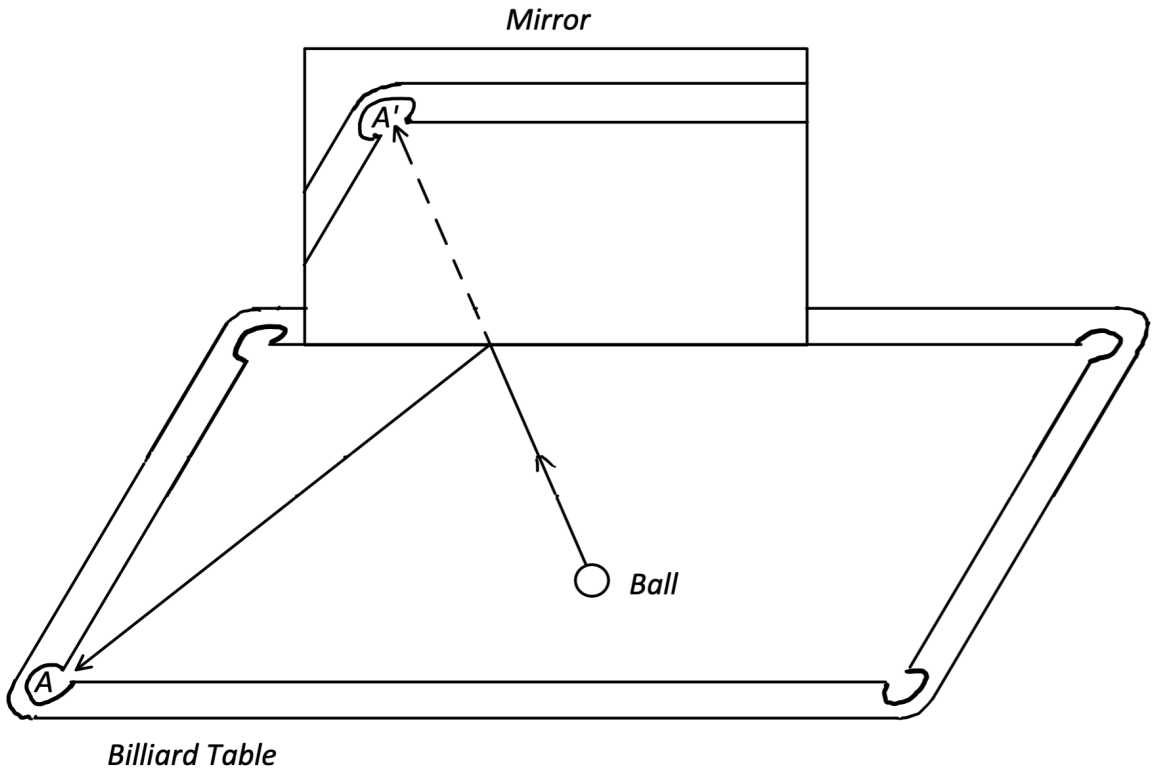
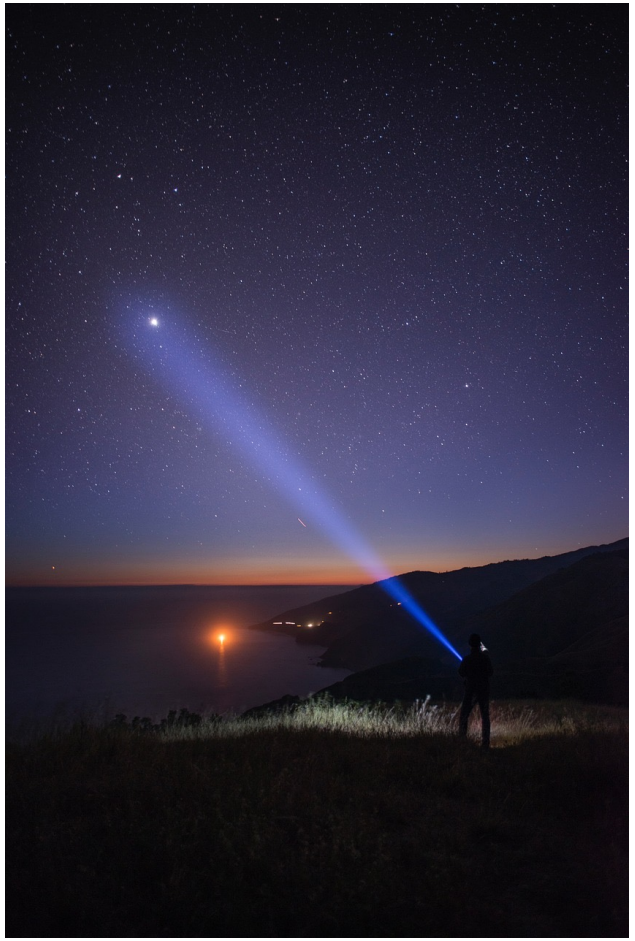


Figure 7: Explanation of the link between the paths of a ray of light and a billiard ball.

Let's extend our "Reflection principle" to solve one more challenging problem



Problem 3

Suppose you have points A(2,4) and B(6,3) on an xy coordinate system, find the length of the shortest path to go from A to B touching first y-axis once, then x-axis once, and finally the vertical line to x-axis y'-axis once, see figure 9 below.

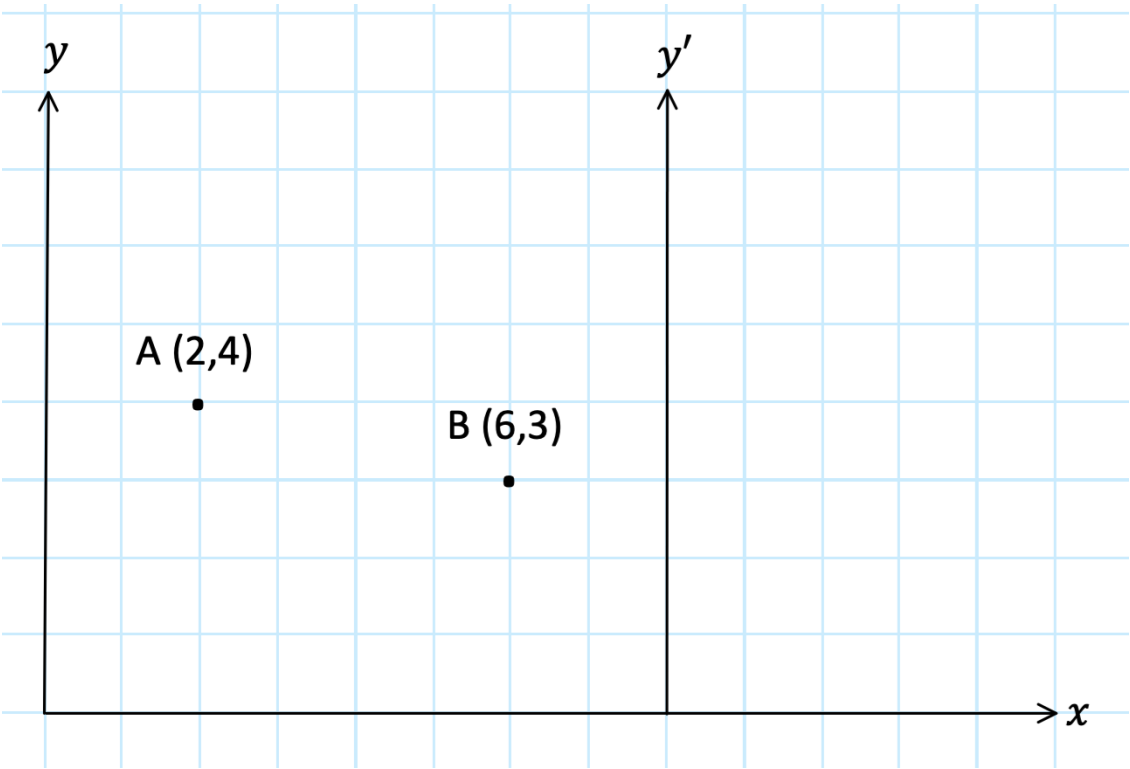


Figure 8: Second extension of our basic problem.

Using our "Reflection Principle", we may ideally start at A' and end at B', which are the reflections of points A and B about y-axis and y'-axis respectively, this in turn is equivalent to ideally starting at A' and ending at B'', where B'' is the reflection of B' about the x-axis. The point where $A'B''$ intersects y-axis let's call it C, and the point where $A'B''$ intersects x-axis let's call it D, the point where DB' intersects y'-axis let's call it E, then the path $A \to C \to D \to E \to B$ is the shortest path, to go from A to B touching once y-axis, x-axis and y'-axis respectively see figure 10 below:

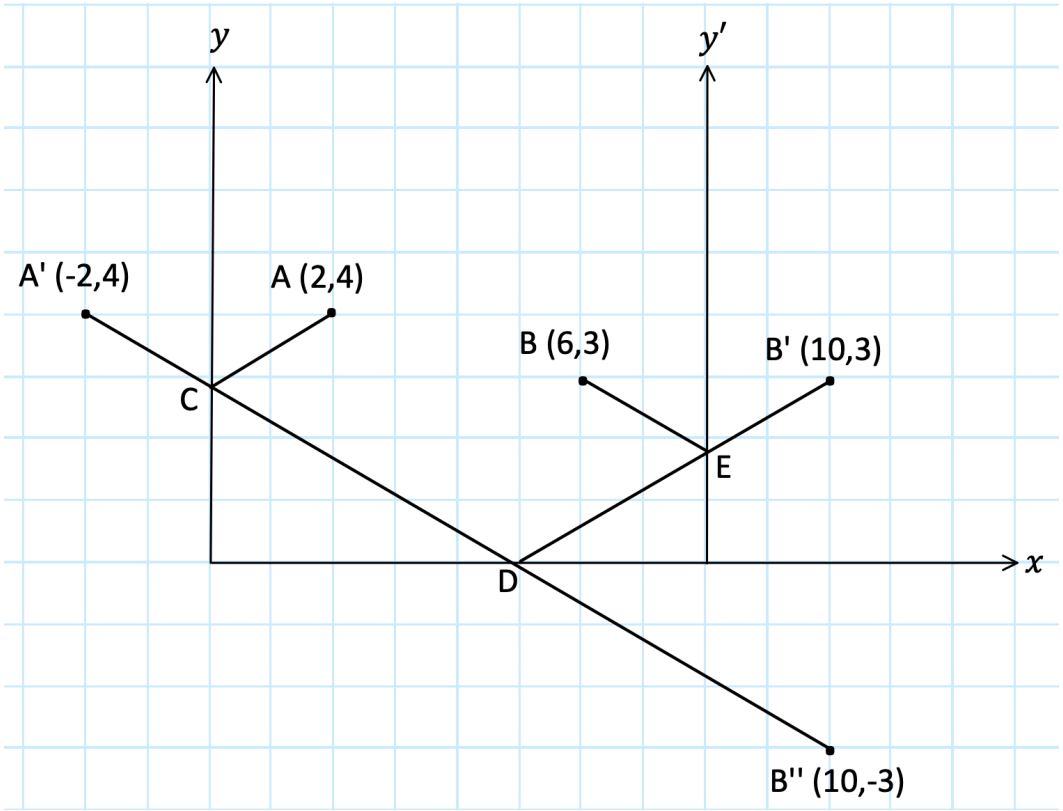


Figure 9: Third application of The Reflection Principle

The equation of the line passing through the points $A'(-2,4)$ and $B''(10,-3)$ is $y = -\frac{7}{12}x + \frac{17}{6}$ with y-intercept $C = \left(0, \frac{17}{6}\right)$ and x-intercept $D = \left(\frac{34}{7}, 0\right)$, the intersection of the line containing the points D and B' $y = \frac{7}{12}x - \frac{17}{6}$ with the vertical line $x = 8$ is the point $E = \left(8, \frac{11}{6}\right)$ this determines the shortest path, the length of the shortest path $A \to C \to D \to E \to B$ can be calculated using the distance formula for the points A' and B'' as shown below:

$$\sqrt{(10-(-2))^2 + (-3-4)^2} = 13.89\ units.$$

Thanks to the Reflection Principle we solved problem 3!

CONCLUSION

The methods shown to solve the problems presented without the use of Calculus, using “The Reflection Principle” have the following Educational and Computational overall advantages:

- All our solutions for problems 1, 2, and 3 using “The Reflection Principle” are constructive, which means that we can geometrically draw the “shortest paths” using only straight edge and compass, this could be the basis for in-class activities, which ultimately, would give our students a “real-world sense” of what the solutions to the “shortest path” problems should be.

- The lengths of the “shortest paths” shown in problems 1, 2, and 3 are similar to the paths that a beam of light from a flashlight would follow if we considered the horizontal and vertical lines as mirrors. This would allow students to confirm a principle that comes from physics, which states that light, in our universe, travels following “the shortest path”

- Our method, based on “The Reflection Principle”, admits a natural extension to solve more challenging problems related to finding “shortest paths” subject to given constraints, this was shown going gradually through problems 1, 2, and 3.

- When solving Applied Optimization Problems like problem 1, 2 and 3 our method based on “The Reflection Principle” uncovers properties, patterns, and associations with real-life events, which are overseen by students when they “blindly” follow the algorithm: “take derivative, set it equal to zero, and solve for the variable”.

Thanks!

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Reference:

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