Taylor & Francis
Taylor & Francis Group

The American Mathematical Monthly

ISSN: (Print) (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

Problems and Solutions

Daniel H. Ullman Edited by, Daniel J. Velleman, Stan Wagon, Douglas B. West & with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by August 31, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12384. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a tetrahedron with AD, BD, and CD mutually perpendicular. Let O and R be the circumcenter and circumradius, respectively, of triangle ABC. Prove $AD^2 + BD^2 + CD^2 + OD^2 = 5R^2$.

12385. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Let n be a positive integer. Prove

$$\sum_{1\le i\le k\le n} \frac{(-2)^k}{k+1}\binom{n}{k}\binom{k}{i}^{-1} = \frac{(-1)^n - 1}{2n}.$$

12386. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Call a permutation $x_0, \dots, x_{n-1}$ of $\{0, 1, \dots, n-1\}$ an *all-interval n-tone row* if the values $x_i - x_{i-1}$ are distinct modulo n for $1 \le i \le n-1$. This requires $x_{n-1} - x_0 \equiv n(n-1)/2 \pmod{n}$. Hence n is even (since $x_{n-1} \ne x_0$) and $|x_{n-1} - x_0| = n/2$. Let T_n be the set of all-interval n-tone rows. When $x \in T_n$ and c and d are integers with c relatively prime to n, let $cx + d$ be the permutation $y_0, \dots, y_{n-1}$ in T_n with $y_k = cx_k + d \pmod{n}$.

(a) For $x \in T_n$, let x^R be the permutation $x_{n-1}, \dots, x_0$, the reverse of x. Note that x^R lies in T_n. Prove that if $x^R = cx + d$, then $c \equiv 1 \pmod{n}$.

(b) For $x \in T_n$, let q be the unique index with $x_q - x_{q-1} \equiv n/2 \pmod{n}$, and let x^Q be the permutation $x_q, x_{q+1}, \dots, x_{n-1}, x_0, x_1, \dots, x_{q-1}$, a rotation of x. Note that x^Q also lies in T_n. Prove that if $x^Q = cx + d$, then $c \equiv -1 \pmod{n}$.

12387. *Proposed by Baris Koyuncu, ENKA Schools, Istanbul, Turkey.* Let a and n be integers greater than 1. For which polynomials $P(x)$ with integer coefficients are there only finitely many primes p such that p divides $P\left(a^{n^k}\right)$ for some positive integer k?

12388. *Proposed by Antonio Garcia, Strasbourg, France.* Let α be a real number. Evaluate

$$\int_0^\infty \frac{(\ln x)^2 \arctan(x)}{1 - 2(\cos\alpha)x + x^2}dx.$$

doi.org/10.1080/00029890.2023.2171615

12389. *Proposed by George Stoica, Saint John, NB, Canada.* Let $f(x) = \sum_{n=1}^{\infty} |\sin(nx)|/n^2$. Prove $\lim_{x\to 0^+} f(x)/(x \ln x) = -1$.

12390. *Proposed by Michael Goldenberg, Reisterstown, MD, and Mark Kaplan, University of Maryland Global Campus, Adelphi, MD.* Let M be the centroid of $\triangle ABC$, and let E be the *Steiner ellipse* of the triangle, which is the unique ellipse E centered at M and passing through A, B, and C.

(a) Show that there are unique ellipses E_B and E_C passing through M with E_B tangent to AB at A and BC at C and with E_C tangent to AC at A and BC at B. Show that E_B and E_C are congruent to E.

(b) Let L be the line through the midpoints W_1 and W_2 of AB and AC, respectively. Let the intersection points of L with E_C be X_1 and X_2, with E be Y_1 and Y_2, and with E_B be Z_1 and Z_2, with subscripts in each case indicating points in the same order along L as W_1 and W_2. Prove

$$\frac{X_1Y_1}{Y_1W_1} = \frac{Y_1W_1}{W_1Z_1} = \frac{W_1Z_1}{Z_1X_2} = \frac{X_2W_2}{Z_1X_2} = \frac{W_2Y_2}{X_2W_2} = \frac{Y_2Z_2}{W_2Y_2} = \frac{1+\sqrt{5}}{2}.$$

SOLUTIONS

The Laplace Transform Simplifies an Integral

12260 [2021, 563]. *Proposed by Seán M. Stewart, Bomaderry, Australia.* Prove

$$\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3}\,dx = \frac{1}{2} - \log 2.$$

Solution by Tewodoros Amdeberham, Tulane University, New Orleans, LA, and Akalu Tefera, Grand Valley State University, Allendale, MI. The Laplace transform $\mathcal{L}$ defined by $\mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$ has the property

$$\int_0^\infty f(x)g(x)\,dx = \int_0^\infty \mathcal{L}[f](s)\cdot\mathcal{L}^{-1}[g](s)\,ds.$$

Applying this with $f(x) = \sin^2 x - x\sin x = 1/2 - (1/2)\cos(2x) - x\sin x$ and $g(x) = 1/x^3$ leads to

$$\begin{aligned}
\int_0^\infty \frac{\sin^2 x - x\sin x}{x^3}\,dx &= \int_0^\infty \mathcal{L}\left[\frac{1}{2} - \frac{1}{2}\cos(2x) - x\sin x\right](s)\cdot\mathcal{L}^{-1}\left[\frac{1}{x^3}\right](s)\,ds\\
&= \int_0^\infty \left(\frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+4} - \frac{2s}{(s^2+1)^2}\right)\cdot\frac{s^2}{2}\,ds\\
&= \int_0^\infty \frac{s}{s^2+4} - \frac{s}{s^2+1} + \frac{s}{(s^2+1)^2}\,ds\\
&= \left[\frac{\log(s^2+4) - \log(s^2+1)}{2} - \frac{1}{2(s^2+1)}\right]_0^\infty = \frac{1}{2} - \log 2.
\end{aligned}$$

Also solved by U. Abel & V. Kushnirevych (Germany), K. F. Andersen (Canada), M. Bataille (France), A. Berkane (Algeria), G. E. Bilodeau, K. N. Boyadzhiev, P. Bracken, B. Bradie, A. C. Castrillón, H. Chen, C. Degenkolb, A. De la Fuente, H. Y. Far, G. Fera (Italy), A. Garcia (France), M. L. Glasser, R. Gordon, H. Grandmontagne (France), G. C. Greubel, N. Grivaux (France), P. Haggstrom (Australia), L. Han (US) &

X. Tan (China), D. Henderson, E. A. Herman, N. Hodges (UK), F. Holland (Ireland), W. Janous (Austria), W. P. Johnson, A. M. Karparvar (Iran), O. Kouba (Syria), K.-W. Lau (China), O. P. Lossers (Netherlands), J. Magliano, K. McLenithan, I. Mező (China), M. Omarjee (France), D. Pinchon (France), S. Sharma (India), P. Shi (China), A. Stadler (Switzerland), J. L. Stitt, R. Stong, R. Tauraso (Italy), Y. Tsyban (Saudi Arabia), J. Van Casteren & L. Kempeneers (Belgium), E. I. Verriest, M. Vowe (Switzerland), S. Wagon, T. Wiandt, H. Widmer (Switzerland), M. Wildon (UK), L. Zhou, Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

Counting Equilateral Triangles in Hypercubes

12261 [2021, 563]. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let a_n be the number of equilateral triangles whose vertices are chosen from the vertices of the n-dimensional cube. Compute $\lim_{n\to\infty} na_n/8^n$.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The limit is $1/(3\sqrt{3}\,\pi)$.

Let the n-dimensional hypercube have vertex set $\{0, 1\}^n$. For vertices A, B, C chosen from this set, let I be the set of coordinates where A differs from both B and C, let J be the set of coordinates where B differs from both A and C, and let K be the set of coordinates where C differs from both A and B. Since $\|A-B\|^2 = |I| + |J|$, $\|B-C\|^2 = |J| + |K|$, and $\|C-A\|^2 = |K| + |I|$, the vertices in $\{A, B, C\}$ form an equilateral triangle if and only if $|I| = |J| = |K|$. Conversely, choose a vertex A and three disjoint sets of indices I, J, K, each of positive size k. Define B to differ from A in coordinates $I \cup J$ and C to differ from A in coordinates $I \cup K$. The resulting triangle ABC is equilateral, and each equilateral triangle arises in 3! ways. Thus,

$$a_n = \frac{2^n}{6} \sum_{k=1}^{\lfloor n/3\rfloor} \binom{n}{3k} \frac{(3k)!}{(k!)^3}. \qquad (*)$$

Stirling's formula gives

$$\frac{(3k)!}{(k!)^3} = \frac{\sqrt{3}}{2\pi k} \cdot 3^{3k} \left(1 + O\left(\frac{1}{k}\right)\right),$$

which we can write equivalently as

$$\frac{(3k)!}{(k!)^3} = \frac{3\sqrt{3}}{2\pi(3k+1)} \cdot 3^{3k} \left(1 + O\left(\frac{1}{k}\right)\right).$$

Since $\binom{n}{3k} \le 2^n$ and $(3k)!/(k!)^3 \le 3^{3k}$, any term in the sum $(*)$ with $k < n/6$ contributes less than $2^n \cdot 2^n \cdot 3^{n/2}$ to a_n. This value, which simplifies to $(4\sqrt{3})^n$, is $o(8^n)$. Therefore, in computing $\lim_{n\to\infty} na_n/8^n$, the sum of the estimates has relative error $O(1/n)$. Also, starting the sum at $k = 0$ has no impact on the limit. Thus

$$\begin{aligned}\frac{na_n}{8^n} &= \frac{(n+1)a_n}{8^n}\left(1 + O\left(\frac{1}{n}\right)\right) = \frac{\sqrt{3}}{4^{n+1}\pi}\left(\sum_{k=0}^{\lfloor n/3\rfloor} \frac{n+1}{3k+1}\binom{n}{3k} 3^{3k}\right)\left(1 + O\left(\frac{1}{n}\right)\right)\\ &= \frac{1}{4^{n+1}\sqrt{3}\,\pi}\left(\sum_{k=0}^{\lfloor n/3\rfloor} \binom{n+1}{3k+1} 3^{3k+1}\right)\left(1 + O\left(\frac{1}{n}\right)\right).\end{aligned}$$

Letting $\omega = e^{2\pi i/3}$ and using $|3\omega + 1| = |3\omega^{-1} + 1| = \sqrt{7} < 4$, it follows that

$$\begin{aligned}\frac{na_n}{8^n} &= \frac{1}{4^{n+1}\sqrt{3}\,\pi} \cdot \frac{(3+1)^{n+1} + \omega^{-1}(3\omega+1)^{n+1} + \omega(3\omega^{-1}+1)^{n+1}}{3}\left(1 + O\left(\frac{1}{n}\right)\right)\\ &= \frac{1}{3\sqrt{3}\,\pi}\left(1 + O\left(\frac{1}{n}\right)\right).\end{aligned}$$

Therefore, the requested limit is $1/(3\sqrt{3}\,\pi)$.

Also solved by U. Abel & V. Kushnirevych (Germany), H. Chen (China), H. Chen (US), R. Dempsey, G. Fera & G. Tescaro (Italy), N. Hodges (UK), M. Omarjee (France), D. Pinchon (France), R. Tauraso (Italy), L. Zhou, and the proposer.

A Trigonometric Generating Function

12262 [2021, 563]. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.* For a nonnegative integer m, let

$$A_m = \sum_{k=0}^{\infty}\left(\frac{1}{(6k+1)^{2m+1}} - \frac{1}{(6k+5)^{2m+1}}\right).$$

Prove $A_0 = \pi\sqrt{3}/6$ and, for $m \geq 1$,

$$2A_m + \sum_{n=1}^{m}\frac{(-1)^n\pi^{2n}}{(2n)!}A_{m-n} = \frac{(-1)^m(4^m+1)\sqrt{3}}{2(2m)!}\left(\frac{\pi}{3}\right)^{2m+1}.$$

Solution by Omran Kouba, Higher Institute for Applied Science and Technology, Damascus, Syria. The sequence $(A_m)_{m\geq 0}$ is bounded, so for $x \in (-1, 1)$ we may define

$$\begin{aligned}F(x) &= \sum_{m=0}^{\infty} A_m x^{2m} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{x^{2m}}{(6k+1)^{2m+1}} - \frac{x^{2m}}{(6k+5)^{2m+1}}\right)\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\left(\frac{x^{2m}}{(6k+1)^{2m+1}} - \frac{x^{2m}}{(6k+5)^{2m+1}}\right)\\ &= \sum_{k=0}^{\infty}\left(\frac{6k+1}{(6k+1)^2-x^2} - \frac{6k+5}{(6k+5)^2-x^2}\right).\end{aligned}$$

Setting $\alpha = (1+x)/6$ and $\beta = (1-x)/6$, we have

$$\begin{aligned}&\frac{6k+1}{(6k+1)^2-x^2} - \frac{6k+5}{(6k+5)^2-x^2}\\ &\qquad = \frac{1}{2}\left(\frac{1}{6k+1+x} + \frac{1}{6k+1-x} - \frac{1}{6k+5+x} - \frac{1}{6k+5-x}\right)\\ &\qquad = \frac{1}{12}\left(\frac{1}{\alpha+k} + \frac{1}{\beta+k} + \frac{1}{\beta-k-1} + \frac{1}{\alpha-k-1}\right).\end{aligned}$$

Next we use the partial fraction expansion of the cotangent, which is

$$\pi\cot(\pi z) = \sum_{k=0}^{\infty}\left(\frac{1}{z+k} + \frac{1}{z-k-1}\right),$$

when z is not an integer. Applying this with $z = \alpha$ and $z = \beta$ gives

$$\begin{aligned}F(x) &= \frac{\pi}{12}\big(\cot(\pi\alpha) + \cot(\pi\beta)\big) = \frac{\pi}{12}\cdot\frac{\sin\big(\pi(\alpha+\beta)\big)}{\sin(\pi\alpha)\sin(\pi\beta)}\\ &= \frac{\pi}{6}\cdot\frac{\sin(\pi(\alpha+\beta))}{\cos\big(\pi(\alpha-\beta)\big) - \cos\big(\pi(\alpha+\beta)\big)} = \frac{\pi}{6}\cdot\frac{\sin(\pi/3)}{\cos(\pi x/3) - \cos(\pi/3)}\\ &= \frac{\pi\sqrt{3}}{6}\cdot\frac{1}{2\cos(\pi x/3) - 1}.\end{aligned}$$

From $(\cos(2\theta)+\cos\theta)(2\cos\theta-1)=\cos(3\theta)+1$, with $\theta=\pi x/3$, we conclude

$$\big(1+\cos(\pi x)\big)F(x)=\frac{\pi\sqrt{3}}{6}\left(\cos\left(\frac{2\pi x}{3}\right)+\cos\left(\frac{\pi x}{3}\right)\right),$$

and hence

$$\left(2+\sum_{n=1}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}x^{2n}\right)\sum_{n=0}^{\infty}A_nx^{2n}=\frac{\pi\sqrt{3}}{6}\sum_{m=0}^{\infty}\frac{(-1)^m(4^m+1)\pi^{2m}}{3^{2m}(2m)!}x^{2m}.$$

Comparing the coefficients of x^{2m} on both sides, we get $A_0=\pi\sqrt{3}/6$ and, for $m\geq 1$,

$$2A_m+\sum_{n=1}^{m}\frac{(-1)^n\pi^{2n}}{(2n)!}A_{m-n}=\frac{(-1)^m(4^m+1)\sqrt{3}}{2(2m)!}\left(\frac{\pi}{3}\right)^{2m+1},$$

as desired.

Editorial comment. Omran Kouba also noted that by using

$$\left(2\cos\left(\frac{\pi x}{3}\right)-1\right)F(x)=\frac{\pi\sqrt{3}}{6},$$

we obtain the alternative recurrence

$$A_m=\sum_{n=1}^{m}\frac{2(-1)^{n-1}}{(2n)!}\left(\frac{\pi}{3}\right)^{2n}A_{m-n}.$$

Also solved by K. F. Andersen (Canada), P. Bracken, H. Chen, G. Fera (Italy), M. L. Glasser, G. C. Greubel, E. A. Herman, N. Hodges (UK), O. P. Lossers (Netherlands), K. Nelson, A. Stadler (Switzerland), M. Štofka (Slovakia), R. Tauraso (Italy), and the proposer.

A Concurrency from A Conic Inscribed in A Triangle

12263 [2021, 564]. *Proposed by Dong Luu, Hanoi National University of Education, Hanoi, Vietnam.* In triangle ABC, let D, E, and F be the points at which the incircle of ABC touches the sides BC, CA, and AB, respectively. Let D', E', and F' be three other points on the incircle with E' and F' on the minor arc EF and D' on the major arc EF and such that AD', BE', and CF' are concurrent. Let X, Y, and Z be the intersections of lines EF and $E'F'$, lines FD and $F'D'$, and lines DE and $D'E'$, respectively. Prove that AX, BY, and CZ are either concurrent or parallel.

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, Netherlands. It is well known that AD, BE, and CF intersect at a point G, the Gergonne point of $\triangle ABC$. We choose homogeneous coordinates such that $A=(1:0:0)$, $B=(0:1:0)$, $C=(0:0:1)$, and $G=(1:1:1)$. It follows that $D=(0:1:1)$, $E=(1:0:1)$, and $F=(1:1:0)$, and the equation of the incircle is $x^2+y^2+z^2-2xy-2xz-2yz=0$.

Since the point of intersection of the lines AD', BE', and CF' lies in the interior of $\triangle ABC$, we can take its coordinates to be $(a^2:b^2:c^2)$, with $a,b,c>0$. This gives $D'=(x:b^2:c^2)$ for some x satisfying the quadratic equation

$$x^2+b^4+c^4-2xb^2-2xc^2-2b^2c^2=0.$$

Of its two solutions $x=(b-c)^2$ and $x=(b+c)^2$, we must choose $x=(b-c)^2$ for D' to be on the major arc EF. Note that since $D\neq D'$, we have $b\neq c$. In the same way we find $E' = (a^2 : (c-a)^2 : c^2)$ and $F' = (a^2 : b^2 : (a-b)^2)$, and a, b, and c are distinct. A somewhat tedious but elementary computation gives

$$X = (a(c-b) : b(c-a) : c(a-b)),$$
$$Y = (a(b-c) : b(a-c) : c(a-b)),$$
$$Z = (a(b-c) : b(c-a) : c(b-a)),$$

so the lines AX, BY, and CZ intersect at the point $(a(b-c) : b(c-a) : c(a-b))$.

Editorial comment. Lossers observed that the solution above works if the incircle is replaced with any ellipse tangent to the sides of the triangle. Li Zhou generalized the problem further by showing that the result holds for any conic tangent to the lines containing the sides of the triangle, with suitable adjustments to the restrictions on the positions of D', E', and F'.

Also solved by L. Zhou and the proposer.

Irreducible Polynomials in Two Variables

12264 [2021, 564]. *Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran.* Let P_d be the set of all polynomials of the form $\sum_{0\le i,j\le d} a_{i,j}x^i y^j$ with $a_{i,j} \in \{1,-1\}$ for all i and j. Prove that there is a positive integer d such that more than 99 percent of the elements of P_d are irreducible in the ring of polynomials with integer coefficients.

Solution by Richard Stong, Center for Communications Research, San Diego, CA. The number 2 is a primitive root modulo the prime p when the smallest value of m such that p divides $2^m - 1$ is $p-1$. Hence the field $\mathbb{F}_{2^{p-1}}$ is the extension of $\mathbb{F}_2$ of lowest degree that contains a primitive pth root of unity modulo 2. It follows that the minimal polynomial of any primitive pth root of unity modulo 2 has degree at least $p-1$. Since the primitive pth roots of unity are the roots of the polynomial $(x^p-1)/(x-1)$ (which equals $x^{p-1} + \cdots + x + 1$ and has degree $p-1$) it follows that this polynomial is irreducible modulo 2. Thus all polynomials of the form $a_0 + a_1x + \cdots + a_{p-1}x^{p-1}$ with all $a_i \in \{-1, 1\}$ (or indeed with all a_i odd) are irreducible over $\mathbb{Z}$.

If $\sum_{0\le i,j\le p-1} a_{i,j}x^i y^j \in P_{p-1}$ is reducible, say as $F(x,y)G(x,y)$, then

$$F(x,0)G(x,0) = a_{0,0} + a_{1,0}x + \cdots + a_{p-1,0}x^{p-1}.$$

Since this polynomial in x is irreducible, $F(x,0)$ or $G(x,0)$ (we may assume $F(x,0)$) has degree $p-1$ as a polynomial in x. Looking at the term with highest degree in x in $F(x,y)G(x,y)$, we conclude that $G(x,y)$ is a constant polynomial in x, and hence we can write $G(x,y)$ as $G(y)$. Swapping the roles of x and y, we find symmetrically that (since $G(y)$ cannot be constant), $G(y)$ has degree $p-1$ and $F(x,y)$ is constant in y, so we write it as $F(x)$. Thus all reducible polynomials in P_{p-1} have the form $F(x)G(y)$. Since $F(0)G(0) = \pm 1$, we conclude $F(0), G(0) \in \{-1, 1\}$, Looking at the terms with degree 0 in x and y yields that all coefficients of $F(x)$ are in $\{1, -1\}$.

Finally, there are 2^p choices for each of F and G, but this double counts the product FG as the product $(-F)(-G)$. Thus there are exactly 2^{2p-1} reducible polynomials in P_{p-1}.

In particular, taking $p = 5$ and noting that 2 is a primitive root modulo 5, we see that only 2^9 of the 2^{25} elements of P_4 are reducible, which is less than 1% of the total number of polynomials in P_4. The fraction only decreases as p increases.

Also solved by S. M. Gagola Jr., O. P. Lossers (Netherlands), D. Pinchon (France), and the proposer.

Combining the Cauchy–Schwarz and AM–GM Inequalities

12267 [2021, 658]. *Proposed by Michel Bataille, Rouen, France.* Let x, y, and z be nonnegative real numbers such that $x + y + z = 1$. Prove

$$(1-x)\sqrt{x(1-y)(1-z)} + (1-y)\sqrt{y(1-z)(1-x)} + (1-z)\sqrt{z(1-x)(1-y)} \geq 4\sqrt{xyz}.$$

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. It is clear that the required inequality holds if any of x, y, or z is zero; it is an equality if two of them are zero. Now suppose that x, y, and z are all positive. Dividing by $\sqrt{xyz}$ and using the fact that $x + y + z = 1$, we see that the inequality is equivalent to

$$\frac{(y+z)\sqrt{(x+z)(x+y)}}{\sqrt{yz}} + \frac{(x+z)\sqrt{(x+y)(y+z)}}{\sqrt{xz}} + \frac{(x+y)\sqrt{(y+z)(x+z)}}{\sqrt{xy}} \geq 4.$$

The Cauchy–Schwarz inequality gives $\sqrt{(x+z)(x+y)} \geq x + \sqrt{yz}$, and by the AM–GM inequality, $y + z \geq 2\sqrt{yz}$. Applying these, we obtain

$$\frac{(y+z)\sqrt{(x+z)(x+y)}}{\sqrt{yz}} \geq \frac{(y+z)(x+\sqrt{yz})}{\sqrt{yz}} = \frac{(y+z)x}{\sqrt{yz}} + y + z \geq 2x + y + z = x + 1.$$

Combining this with similar inequalities for the other two terms, we get

$$\frac{(y+z)\sqrt{(x+z)(x+y)}}{\sqrt{yz}} + \frac{(x+z)\sqrt{(x+y)(y+z)}}{\sqrt{xz}} + \frac{(x+y)\sqrt{(y+z)(x+z)}}{\sqrt{xy}}$$
$$\geq (x+1) + (y+1) + (z+1) = 4,$$

as required. When x, y, and z are positive, equality holds only if $x = y = z = 1/3$.

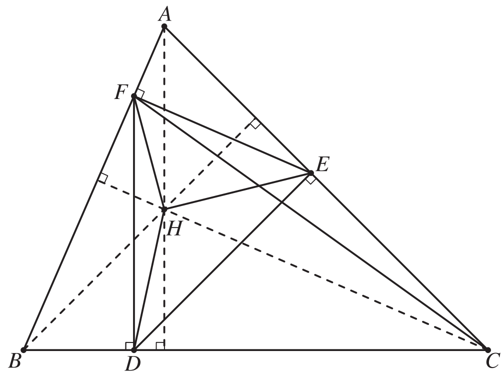
Also solved by A. Alt, F. R. Ataev (Uzbekistan), A. Berkane (Algeria), P. Bracken, H. Chen (China), H. Chen, C. Chiser (Romania), N. S. Dasireddy (India), M. Dinča (Romania), H. Y. Far, G. Fera (Italy), A. Garcia (France), O. Geupel (Germany), P. Haggstrom (Australia), D. Henderson, N. Hodges (UK), F. Holland (Ireland), E. J. Ionaşcu, W. Janous (Austria), A. M. Karparvar (Iran), P. Khalili, K. T. L. Koo (Hong Kong), O. Kouba (Syria), K.-W. Lau (Hong Kong), S. Lee (Korea), O. P. Lossers (Netherlands), J. F. Loverde, A. Mhanna (Lebanon), M. Reid, V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), J. F. Gonzalez & F. A. Velandia (Colombia), M. Vowe (Switzerland), J. Vukmirović (Serbia), H. Widmer (Switzerland), L. Wimmer (Germany), L. Zhou, UM6P MathClub (Morocco), and the proposer.

A Triangle Inscribed in a Similar Triangle

12269 [2021, 659]. *Proposed by Mehmet Şahin and Ali Can Güllü, Ankara, Turkey.* Let ABC be an acute triangle. Suppose that D, E, and F are points on sides BC, CA, and AB, respectively, such that FD is perpendicular to BC, DE is perpendicular to CA, and EF is perpendicular to AB. Prove

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = 1.$$

Solution I by Michael Reid, University of Central Florida, Orlando, FL. For a polygon $PQ \cdots Z$, let $(PQ \cdots Z)$ denote its area. Let H be the orthocenter of $\triangle ABC$. Since the triangle is acute, H lies in its interior. Both CH and EF are perpendicular to AB, so they are parallel, and therefore $(CEF) = (HEF)$. Thus



$$\frac{AF}{AB} = \frac{(AFC)}{(ABC)} = \frac{(AFE) + (CEF)}{(ABC)} = \frac{(AFE) + (HEF)}{(ABC)} = \frac{(HEAF)}{(ABC)}.$$

Similarly, $BD/BC = (HFBD)/(ABC)$ and $CE/CA = (HDCE)/(ABC)$, so

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = \frac{(HEAF) + (HFBD) + (HDCE)}{(ABC)} = \frac{(ABC)}{(ABC)} = 1.$$

Solution II by Li Zhou, Polk State College, Winter Haven, FL. By Miquel's theorem, the circumcircles of triangles AFE, BDF, and CED concur at a point, the Miquel point M. Note that since $\angle AFE$ is a right angle, AE is a diameter of the circumcircle of $\triangle AFE$, and therefore $\angle AME$ is also a right angle. Similarly, $\angle BMF$ and $\angle CMD$ are right angles.

Since $\angle MFE$ and $\angle MAE$ are subtended by the same arc of the circumcircle of $\triangle AFE$, they are equal. Similarly, $\angle MED = \angle MCD$ and $\angle MDF = \angle MBF$. Also, $\angle MAE = \angle MED$, since both are complementary to $\angle MEA$, and similarly $\angle MCD = \angle MDF$. We conclude that all six of the angles $\angle MFE$, $\angle MAE$, $\angle MED$, $\angle MCD$, $\angle MDF$, and $\angle MBF$ are equal. This means that M is a Brocard point of both $\triangle ABC$ and $\triangle DEF$. Let ω denote the measure of all six angles, which is the Brocard angle. It is well known that $\cot\omega = \cot A + \cot B + \cot C$.

Triangles MEF and MAB are similar, since corresponding sides are perpendicular. Hence $EF/AB = EM/AM$, so

$$\frac{AF}{AB} = \frac{AF}{EF} \cdot \frac{EF}{AB} = \cot A \cdot \frac{EM}{AM} = \cot A \tan\omega.$$

Similarly, $BD/BC = \cot B \tan\omega$ and $CE/CA = \cot C \tan\omega$, so

$$\frac{AF}{AB} + \frac{BD}{BC} + \frac{CE}{CA} = (\cot A + \cot B + \cot C)\tan\omega = \cot\omega\tan\omega = 1.$$

Editorial comment. Several readers noted that the result can be extended to obtuse triangles by allowing one of the points D, E, and F to lie on an extension of a side of $\triangle ABC$ and using signed distances.

It was not required to construct $\triangle DEF$, or even to show that such a triangle exists. However, Solution II shows how to construct the unique such triangle. Let M be the Brocard point of $\triangle ABC$ such that $\angle MAC$, $\angle MBA$, and $\angle MCB$ all have the same measure ω. Triangle DEF is the image of triangle CAB under a rotation of $\pi/2$ radians about M followed by a dilation centered at M with ratio $\tan\omega$.

Also solved by M. Bataille (France), R. B. Campos (Spain), H. Chen (China), C. Chiser (Romania), M. Dincă, G. Fera (Italy), D. Fleischman, K. Gatesman, O. Geupel (Germany), E. A. Herman, N. Hodges (UK),

E. J. Ionaşcu, Y. J. Ionin, W. Janous (Austria), W. Ji (China), M. Goldenberg & M. Kaplan, A. M. Karparvar (Iran), P. Khalili, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), J. McHugh, M. D. Meyerson, J. Minkus, M. R. Modak (India), C. G. Petalas (Greece), C. R. Pranesachar (India), I. Retamoso, V. Schindler (Germany), A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Vowe (Switzerland), J. Vukmirović (Serbia), T. Wiandt, H. Widmer (Switzerland), L. Wimmer (Germany), T. Zvonaru (Romania), Davis Problem Solving Group, Fejéntaláltuka Szeged Problem Solving Group (Hungary), UM6P Math Club (Morocco), and the proposer.

A Refinement of a Putnam Problem

12270 [2021, 659]. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.* Let $a_0 = 1$, and let $a_{n+1} = a_n + e^{-a_n}$ for $n \geq 0$. Show that the sequence whose nth term is $e^{a_n} - n - (1/2)\ln n$ converges.

Solution by Kuldeep Sarma, Tezpur University, Tezpur, India. Define $u_n = e^{a_n}$, and note that $u_{n+1} = u_n e^{1/u_n}$. Since the sequence $\{u_n\}$ is positive and strictly increasing, it must either converge to a positive limit or diverge to $+\infty$. If the sequence converges to L, then the recurrence relation gives $L = Le^{1/L}$, which is impossible; therefore $\lim_{n\to\infty} u_n = +\infty$.

Note that $\lim_{n\to\infty}(u_{n+1} - u_n) = \lim_{n\to\infty} u_n(e^{1/u_n} - 1) = 1$. Therefore, by the Stolz–Cesàro theorem, $\lim_{n\to\infty} u_n/n = 1$. It follows that

$$\lim_{n\to\infty} \frac{u_{n+1} - u_n - 1}{1/n} = \lim_{n\to\infty} \frac{u_n^2(e^{1/u_n} - 1 - 1/u_n)}{u_n/n} = \frac{1/2}{1} = \frac{1}{2}.$$

By the Stolz–Cesàro theorem again,

$$\begin{aligned}\lim_{n\to\infty} \frac{u_n - n}{\ln n} &= \lim_{n\to\infty} \frac{(u_{n+1} - (n+1)) - (u_n - n)}{\ln(n+1) - \ln n} \\ &= \lim_{n\to\infty} \frac{u_{n+1} - u_n - 1}{1/n} \cdot \frac{1/n}{\ln(1+1/n)} = \frac{1}{2} \cdot 1 = \frac{1}{2}.\end{aligned}$$

Combining the recurrence relation for u_n with the Maclaurin series for the exponential function, for $n \geq 1$ we have

$$u_{n+1} = u_n + 1 + \frac{1}{2u_n} + O\left(\frac{1}{u_n^2}\right) = u_n + 1 + \frac{1}{2n} - \frac{u_n - n}{2nu_n} + O\left(\frac{1}{u_n^2}\right).$$

From previous observations, we know that

$$\frac{u_n - n}{2nu_n} \sim \frac{\ln n}{4n^2} \quad \text{and} \quad \frac{1}{u_n^2} \sim \frac{1}{n^2},$$

so

$$u_{n+1} = u_n + 1 + \frac{1}{2n} + O\left(\frac{\ln n}{n^2}\right).$$

Since $\sum_{n=1}^{\infty} \ln n/n^2$ converges, we conclude that $\sum_{n=1}^{N-1}(u_{n+1} - u_n - 1 - 1/(2n))$ converges as $N \to \infty$. For $N \geq 2$,

$$\sum_{n=1}^{N-1}\left(u_{n+1} - u_n - 1 - \frac{1}{2n}\right) = u_N - u_1 - (N-1) - \frac{H_{N-1}}{2},$$

where we write H_k for the kth harmonic number $\sum_{i=1}^{k} 1/i$. Therefore

$$e^{a_N} - N - \frac{1}{2}\ln N = \sum_{n=1}^{N-1}\left(u_{n+1} - u_n - 1 - \frac{1}{2n}\right) + u_1 - 1 - \frac{1}{2N} + \frac{1}{2}(H_N - \ln N).$$

The desired result follows, since $H_N - \ln N \to \gamma$ as $N \to \infty$.

Editorial comment. Several solvers noted similarities between this problem and MONTHLY Problem 11837 [2015, 391; 2017, 91], which asks for a proof that the sequence $\{a_n - \ln n\}$ decreases monotonically to 0. The earlier MONTHLY problem is a refinement of Problem B4 of the 73rd William Lowell Putnam Mathematical Competition, which simply asks whether $\{a_n - \ln n\}$ has a finite limit. Indeed, since $a_n - \ln n = \ln(u_n/n)$, it follows from the above solution that $\lim_{n\to\infty}(a_n - \ln n) = 0$. This solves the Putnam problem and part of the earlier MONTHLY problem.

Also solved by M. Bataille (France), A. Berkane (Algeria), P. Bracken, H. Chen, N. Grivaux (France), X. Tang (China) & L. Han (US), E. A. Herman, N. Hodges (UK), E. J. Ionaşcu, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, O. P. Lossers (Netherlands), S. Omar (Morocco), E. Omey (Belgium), A. Stadler (Switzerland), A. Stenger, R. Stong, R. Tauraso (Italy), J. Vukmirović (Serbia), J. Yan (China), UM6P Math Club (Morocco), and the proposer.

CLASSICS

C14. *Due to Paul Erdős and George Szekeres; suggested by the editors.* Show that no two entries chosen from the interior of any row of Pascal's triangle are relatively prime.

Visiting Every Region on a Sphere Exactly Once

C13. *Due to Leo Moser; suggested by the editors.* Let n be a multiple of 4, and consider an arrangement of n great circles on the sphere, no three concurrent, dividing the sphere into regions. Show that there is no path on the sphere that visits each region once and only once and never passes through an intersection point of two of the great circles.

Solution. The great circles define a graph G: the vertices are the intersection points of the circles, and the edges are the arcs of the circles joining vertices. Let H be the graph of the corresponding map: the vertices are the regions of G, and edges connect adjacent regions across an edge of G. Because any two great circles intersect twice, G has $n(n-1)$ vertices. Because every vertex of G has four neighbors, G has $2n(n-1)$ edges. By Euler's formula $V - E + F = 2$ relating the numbers of vertices, edges, and faces of a connected graph on the sphere, G has $n(n-1)+2$ faces. This is the number of vertices of H and is even.

Since every edge in H crosses a great circle, and every cycle in H must cross each great circle an even number of times to return to the original region, every cycle in H has even length. Hence H is bipartite, meaning that we can color each vertex of H red or blue in such a way that all edges connect a red vertex and a blue vertex.

The regions of G containing diametrically opposite points on the sphere lie on opposite sides of every great circle. Hence every path joining the vertices for these points crosses every great circle an odd number of times. Since n is even, this implies that such a path has even length, so the vertices representing antipodal regions are colored the same. It follows that H has an even number of vertices of each color.

If H has a path that visits each vertex, then H must have the same number of vertices of each color. Since the two color classes have the same even size, the number of vertices in H is a multiple of 4. However, that number is $n(n-1)+2$, which is not divisible by 4.

Editorial comment. This problem appeared in this MONTHLY as problem E788 [1947, 471; 1948, 366] and is due to Leo Moser. There is an essentially unique arrangement of n great circle arcs on a sphere when $n \le 5$, and for $n \in \{2, 3, 5\}$ each of these arrangements does permit a Hamiltonian path, in fact a Hamiltonian circuit. When $n = 6$, some arrangements permit Hamiltonian paths and some do not.

