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Problems and Solutions

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PROBLEMS

Proposals

To be considered for publication, solutions should be received by March 1, 2023.

2151. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Let $ABCD$ and $XYZT$ be two directly similar squares such that A and Y lie on the lines XT and CD, respectively. Let M be the intersection of lines XZ and AC, and let N be the intersection of lines XY and BC. Prove that circumcenter of $\triangle XAC$ lies on the line MN.

2152. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.*

Evaluate

$$\int_0^1 \int_0^1 \frac{dy\,dx}{\sqrt{1-x^2}\sqrt{1-y^2}\,(1+xy)}.$$

2153. *Proposed by Rex H. Wu, New York, NY.*

Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Evaluate the following for $k \geq 0$.

(a) $\displaystyle\sum_{n=0}^{\infty} \arctan \frac{F_{2k}}{F_{2n+1}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \arctan \frac{L_{2k+1}}{L_{2n}}$

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com*. More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2154. *Proposed by the Columbus State University Problem Solving Group, Columbus, GA.*

Let $f(n)$ denote the number of ordered partitions of a positive integer n such that all of the parts are odd. For example, $f(5) = 5$, since 5 can be written as $5, 3+1+1, 1+3+1, 3+1+1$, and $1+1+1+1+1$. Determine $f(n)$.

2155. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let R be a ring with identity and U a subset of the units of R with $|U| = p$, where p is an odd prime. Suppose that for all $a \in R$, there is a $u \in U$ and a $k \in \mathbb{Z}^+$ such that $ua^k = a^{k+1}$. Show that

(a) For all $a \in R$, there is a $u \in U$ such that $ua = a^2$.
(b) The ring R is commutative.

Quickies

1123. *Proposed by George Stoica, Saint John, NB, Canada.*

Given a function $f : \mathbb{R}^\ell \to \mathbb{R}$, we say that f satisfies condition P_n if

$$f\left(\frac{1}{n}\sum_{i=1}^{n} A_i\right) = \frac{1}{n}\sum_{i=1}^{n} f(A_i)$$

for all $A_1, \ldots, A_n \in \mathbb{R}^\ell$. Show that for all $m, n \geq 2$, conditions P_m and P_n are equivalent.

1124. *Proposed by A. Berele and T. Kyle Petersen, DePaul University, Chicago, IL.*

Let $\{F_n\}_{n=1}^\infty$ be the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$. Does there exist an infinite subsequence $\{F_{n_i}\}_{i=1}^\infty$, the sum of whose reciprocals converges to 1?

Solutions

Minimize the length of the tangent segment **October 2021**

2126. *Proposed by M. V. Channakeshava, Bengaluru, India.*

A tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the x-axis and y-axis at the points A and B, respectively. Find the minimum value of AB.

Solution by Kangrae Park (student), Seoul National University, Seoul, Korea.
We may assume that $a, b > 0$ and that the point of tangency $P = (\alpha, \beta)$ lies in the first quadrant. One readily verifies that the tangent line to the ellipse at P is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1.$$

Therefore, A and B are $(a^2/\alpha, 0)$ and $(0, b^2/\beta)$, respectively. Note that

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1$$

since the point P is on the ellipse. Applying the Cauchy-Schwarz inequality with

$$\mathbf{u} = \left(\frac{a^2}{\alpha}, \frac{b^2}{\beta}\right) \quad \text{and} \quad \mathbf{v} = \left(\frac{\alpha}{a}, \frac{\beta}{b}\right),$$

we obtain

$$\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2} = \left(\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right) = (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) \geq (\mathbf{u}\cdot\mathbf{v})^2 = (a+b)^2 .$$

It follows that

$$AB = \sqrt{\frac{a^4}{\alpha^2} + \frac{b^4}{\beta^2}} \geq a + b.$$

This lower bound is attained if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. A straightforward calculation shows that this occurs if and only if

$$\alpha^2 = \frac{a^3}{a+b} \quad \text{and} \quad \beta^2 = \frac{b^3}{a+b}.$$

This gives the esthetically pleasing result that when AB attains its minimum value of $a+b$, we have $PB = a$ and $PA = b$.

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Yagub Aliyev (Azerbaijan), Michel Bataille (France), Bejmanin Bittner, Khristo Boyadzhiev, Paul Bracken, Brian Bradie, Robert Calcaterra, Hongwei Chen, Joowon Chung (South Korea), Robert Doucette, Rob Downes, Eagle Problem Solvers (Georgia Southern University), Habib Y. Far, John Fitch, Dmitry Fleischman, Noah Garson (Canada), Kyle Gatesman, Subhankar Gayen (India), Jan Grzesik, Emmett Hart, Eugene A. Herman, David Huckaby, Tom Jager, Walther Janous (Austria), Mark Kaplan & Michael Goldenberg, Kee-Wai Lau (Hong Kong), Lucas Perry & Alexander Perry, Didier Pinchon (France), Ivan Retamoso, Celia Schacht, Randy Schwartz, Ioannis Sfikas (Greece), Vishwesh Ravi Shrimali (India), Albert Stadler (Switzerland), Seán M. Stewart (Saudi Arabia), David Stone & John Hawkins, Nora Thornber, R. S. Tiberio, Michael Vowe (Switzerland), Lienhard Wimmer (Germany), and the proposer. There were seventeen incomplete or incorrect solutions.

Two idempotent matrices **October 2021**

2127. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA and Roger Horn, Tampa, FL.*

Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ is such that $AB = A$ and $BA = B$. Show that

(a) A and B are idempotent and have the same null space.

(b) If $1 \leq \text{rank } A < n$, then there are infinitely many choices of B that satisfy the hypotheses.

(c) $A = B$ if and only if $A - I$ and $B - I$ have the same null space.

Solution by Michel Bataille, Rouen, France.

(a) The fact that $A^2 = A$ and $B^2 = B$ follows from:

$$A^2 = (AB)A = A(BA) = AB = A, \qquad B^2 = (BA)B = B(AB) = BA = B.$$

In addition, if X is a column vector and $AX = 0$, then $BAX = 0$, that is, $BX = 0$. Thus, $\ker A \subseteq \ker B$. Similarly, if $BX = 0$, then $ABX = 0$. Hence $AX = 0$ so that $\ker B \subseteq \ker A$. We conclude that $\ker A = \ker B$.

(b) Let $r = \text{rank}(A)$. Since A is idempotent, we have $\text{range}(A) \oplus \ker A = \mathbb{C}^n$. Since $AX = X$ if $X \in \text{range}(A)$ and $\dim(\text{range}(A)) = r$, it follows that $A = PJ_rP^{-1}$ for some invertible $n \times n$ matrix P and

$$J_r = \left(\begin{array}{c|c} I_r & O \\ \hline O & O \end{array}\right),$$

where I_r denotes the $r \times r$ unit matrix and O a null matrix of the appropriate size. Consider the matrices $B = PB'P^{-1}$ with

$$B' = \left(\begin{array}{c|c} I_r & O \\ \hline C & O \end{array}\right),$$

where C is an arbitrary $(n-r) \times r$ matrix with complex entries. There are infinitely many such matrices B, and we calculate

$$AB = PJ_rP^{-1}PB'P^{-1} = PJ_rB'P^{-1} = PJ_rP^{-1} = A,$$

and

$$BA = PB'P^{-1}PJ_rP^{-1} = PB'J_rP^{-1} = PB'P^{-1} = B.$$

(c) Clearly, $A - I$ and $B - I$ have the same null space if $A = B$. Conversely, suppose that $\ker(A - I) = \ker(B - I)$. Let X be a column vector. Since $(A - I)A = O$, the vector AX is in $\ker(A - I)$, hence is in $\ker(B - I)$. This means that $(B - I)AX = 0$, that is, $BX = AX$ (since $BA = B$). Since X is arbitrary, we can conclude that $A = B$.

Also solved by Paul Budney, Robert Calcaterra, Hongwei Chen, Robert Doucette, Dmitry Fleischman, Kyle Gatesman, Eugene A. Herman, Tom Jager, Rachel McMullan, Thoriq Muhammad (Indonesia), Didier Pinchon (France), Michael Reid, Randy Schwartz, Omar Sonebi (Morroco), and the proposer. There was one incomplete or incorrect solution.

Two exponential inequalities — October 2021

2128. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $0 < a < b < 1$ and $\epsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1 - b^m)^n < \epsilon$ and $(1 - a^m)^n > 1 - \epsilon$.

Solution by Robert Doucette, McNeese State University, Lake Charles, LA.
It is well known that

$$\lim_{x\to 0}(1 - x)^{1/x} = e^{-1}.$$

Suppose $0 < \alpha < 1$. Then, since $\alpha^x \to 0^+$ as $x \to \infty$,

$$\lim_{x\to\infty}(1 - \alpha^x)^{\alpha^{-x}} = e^{-1}.$$

Hence,

$$\lim_{x\to\infty}(1 - \alpha^x)^{\beta^{-x}} = \lim_{x\to\infty}\left[(1 - \alpha^x)^{\alpha^{-x}}\right]^{(\beta/\alpha)^{-x}} = \begin{cases} 0, & \text{if } 0 < \beta < \alpha < 1 \\ 1, & \text{if } 0 < \alpha < \beta < 1 \end{cases}.$$

Choose c and d such that $0 < a < c < d < b < 1$. Note that $c^{-x} - d^{-x} \to \infty$ as $x \to \infty$.

By the limits established above, there exists a positive integer m such that

$$(1-b^m)^{d^{-m}} < \epsilon, (1-a^m)^{c^{-m}} > 1-\epsilon, \text{ and } c^{-m} - d^{-m} > 1.$$

There also exists a positive integer n such that $d^{-m} < n < c^{-m}$. Therefore,

$$(1-b^m)^n < (1-b^m)^{d^{-m}} < \epsilon \text{ and } (1-a^m)^n > (1-a^m)^{c^{-m}} > 1-\epsilon.$$

Also solved by Levent Batakci, Michel Bataille (France), Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Bruce Burdick, Michael Cohen, Dmitry Fleischman, Kyle Gatesman, Michael Goldenberg & Mark Kaplan, Eugene Herman, Miguel Lerma, Reiner Martin (Germany), Raymond Mortini (France), Michael Nathanson, Moubinool Omajee (France), Didier Pinchon (France), Albert Stadler (Switzerland), Omar Sonebi (Morroco), and the proposer.

Two improper integrals **October 2021**

2129. *Proposed by Vincent Coll and Daniel Conus, Lehigh University, Bethlehem, PA and Lee Whitt, San Diego, CA.*

Determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{2^k}\right) dx$

(b) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{3^k}\right) dx$

Solution by Gerald A. Edgar, Denver, CO.
(a) The integral diverges. For $0 < x < 1$ we have

$$\log\frac{1}{1-x} = \sum_{n=1}^{\infty}\frac{1}{n}x^n = \sum_{k=0}^{\infty}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{n}x^n\right)$$
$$\le \sum_{k=0}^{\infty}\left(\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{2^k}x^{2^k}\right) = \sum_{k=0}^{\infty}\left(\frac{2^k}{2^k}x^{2^k}\right) = \sum_{k=0}^{\infty}x^{2^k}.$$

Therefore,

$$\exp\left(\sum_{k=0}^{\infty}x^{2^k}\right) \ge \frac{1}{1-x}.$$

The integral (a) diverges by comparison with the divergent integral $\int_0^1 dx/(1-x)$.

(b) The integral converges. We will need an estimate for a harmonic sum. The function $1/x$ is decreasing, so for $k \ge 1$

$$\sum_{n=3^{k-1}}^{3^k-1}\frac{1}{n} > \int_{3^{k-1}}^{3^k}\frac{dx}{x} = \log 3.$$

Now, for $0 < x < 1$ we have

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{k=1}^{\infty} \left(\sum_{n=3^{k-1}}^{3^k-1} \frac{1}{n} x^n \right)$$

$$> \sum_{k=1}^{\infty} \left(\sum_{n=3^{k-1}}^{3^k-1} \frac{1}{n} \right) x^{3^k} > \sum_{k=1}^{\infty} (\log 3) x^{3^k}.$$

Let $r = 1/\log 3$, so that $0 < r < 1$. Then

$$r \log \frac{1}{1-x} > \sum_{k=1}^{\infty} x^{3^k},$$

$$\log \frac{1}{(1-x)^r} + 1 > \sum_{k=0}^{\infty} x^{3^k},$$

$$\frac{e}{(1-x)^r} > \exp\left(\sum_{k=0}^{\infty} x^{3^k} \right).$$

The integral (b) converges by comparison with the convergent integral

$$\int_0^1 \frac{e}{(1-x)^r}\, dx.$$

Editor's Note. A more detailed analysis shows that

$$\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{\alpha^k} \right) dx$$

converges if $\alpha > e$ and diverges if $1 \le \alpha \le e$.

Also solved by Michael Bataille (France), Robert Calcaterra, Dmitry Fleischman, Eugene A. Herman, Walther Janous (Austria), Albert Natian, Moubinool Omarjee (France), Didier Pinchon (France), Albert Stadler (Switzerland), and the proposers. There was one incomplete or incorrect solution.

When does the circumcenter lie on the incircle? October 2021

2130. *Proposed by Florin Stanescu, Şerban Cioculescu School, Găeşti, Romania.*

Given the acute $\triangle ABC$, let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Choose $P, R \in \overleftrightarrow{AB}$, $S, T \in \overleftrightarrow{BC}$, $Q, U \in \overleftrightarrow{AC}$ so that

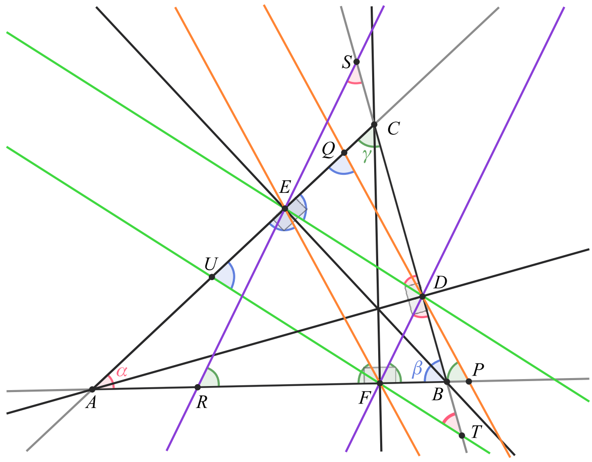
$$D \in \overleftrightarrow{PQ},\ E \in \overleftrightarrow{RS},\ F \in \overleftrightarrow{TU} \ \text{ and } \ \overleftrightarrow{PQ} \parallel \overleftrightarrow{EF},\ \overleftrightarrow{RS} \parallel \overleftrightarrow{DF},\ \overleftrightarrow{TU} \parallel \overleftrightarrow{DE}.$$

Show that

$$\frac{PQ + RS - TU}{AB} + \frac{RS + TU - PQ}{BC} + \frac{TU + PQ - RS}{AC} = 2\sqrt{2}$$

if and only if the circumcenter of $\triangle ABC$ lies on the incircle of $\triangle ABC$.

Solution by the Fejéntaláltuka Szeged Problem Solving Group, University of Szeged, Szeged, Hungary.



Let O and I be the circumcenter and the incenter of $\triangle ABC$. Then Euler's theorem states that $OI^2 = R(R-2r)$, where R and r are the circumradius and the inradius of the triangle, respectively. Now O lies on the incircle if and only if $R(R-2r) = r^2$, which is equivalent to $\left(\frac{r}{R}\right)^2 + 2\frac{r}{R} - 1 = 0$. Therefore, $\frac{r}{R} = \sqrt{2} - 1$ since $\frac{r}{R} > 0$. Since $\cos\alpha + \cos\beta + \cos\gamma = 1 + \frac{r}{R}$ in any triangle, we can reduce the original condition to $\cos\alpha + \cos\beta + \cos\gamma = \sqrt{2}$ where α, β and γ are the angles of $\triangle ABC$.

We have

$$\begin{aligned} DE^2 &\overset{(1)}{=} CD^2 + CE^2 - 2CD \cdot CE\cos\gamma \\ &\overset{(2)}{=} (CA\cos\gamma)^2 + (BC\cos\gamma)^2 - 2(CA\cos\gamma)(BC\cos\gamma)\cos\gamma \\ &= (CA^2 + BC^2 - 2CA \cdot BC\cos\gamma)\cos^2\gamma \overset{(3)}{=} AB^2\cos^2\gamma, \end{aligned}$$

where (1) and (3) are the result of the law of cosines applied to $\triangle CDE$ and $\triangle ABC$, respectively, and (2) follows from the fact that CD and CE are altitudes. Since $\triangle ABC$ is acute, $\cos\alpha > 0$, so

$$DE = AB\cos\gamma, \text{ and similarly } EF = BC\cos\alpha \text{ and } FD = CA\cos\beta. \qquad (1)$$

Because $\angle BFC$ and $\angle BEC$ are right angles, E and F lie on the circle with diameter BC, thus $BCEF$ is a cyclic quadrilateral. Hence, $m\angle EFA = 180^\circ - m\angle BFE = m\angle ECB = \gamma$ and $m\angle AEF = 180^\circ - m\angle FEC = m\angle CBF = \beta$. We can similarly see that $m\angle FDB = m\angle CDE = \alpha$, $m\angle DEC = \beta$ and $m\angle BFD = \gamma$. Since $PQ \parallel EF$, $RS \parallel FD$ and $TU \parallel DE$ we have

$$\begin{aligned} m\angle RSB = m\angle FDB = \alpha = m\angle CDE = m\angle CTU, \\ m\angle AQP = m\angle AEF = \beta = m\angle DEC = m\angle TUC, \\ m\angle BRS = m\angle BFD = \gamma = m\angle EFA = m\angle QPA. \end{aligned}$$

Therefore, the following triangles are all isosceles (because they all have two congruent angles): $\triangle DQE$, $\triangle EDS$, $\triangle ERF$, $\triangle FEU$, $\triangle FTD$, and $\triangle DFP$. Therefore,

$$DQ = DE = ES, RE = EF = FU, \text{ and } TF = FD = PD,$$

which (by (1)) leads to

$$PQ = PD + DQ = FD + DE = CA\cos\beta + AB\cos\gamma,$$
$$RS = RE + ES = EF + DE = BC\cos\alpha + AB\cos\gamma,$$
$$TU = TF + FU = FD + EF = CA\cos\beta + BC\cos\alpha.$$

Substituting these into our original statement, we get that

$$\frac{PQ + RS - TU}{AB} + \frac{RS + TU - PQ}{BC} + \frac{TU + PQ - RS}{CA} = 2(\cos\gamma + \cos\alpha + \cos\beta).$$

In the first paragraph, we showed that the right side of the last equation equals $2\sqrt{2}$ if and only if the circumcenter lies on the incircle, which is exactly what we wanted to prove.

Also solved by Michel Bataille (France), Kyle Gatesman, Volkhard Schindler (Germany), Albert Stadler (Switzerland), and the proposer.

Answers

Solutions to the Quickies from page 407.

A1123. We will need the fact that if f satisfies P_2, then

$$f\left(\frac{n}{n+1}A_1 + \frac{1}{n+1}A_2\right) = \frac{n}{n+1}f(A_1) + \frac{1}{n+1}(A_2). \tag{1}$$

We proceed by induction. When $n = 1$ this is just condition P_2. Let

$$X = \frac{n+1}{n+2}A_1 + \frac{1}{n+2}A_2 \quad \text{and} \quad Y = \frac{1}{n+2}A_1 + \frac{n+1}{n+2}A_2.$$

We have

$$X = \frac{n}{n+1}A_1 + \frac{1}{n+1}Y \quad \text{and} \quad Y = \frac{1}{n+1}X + \frac{n}{n+1}A_2,$$

so, by the induction hypothesis,

$$f(X) = \frac{n}{n+1}f(A_1) + \frac{1}{n+1}f(Y) \quad \text{and} \quad f(Y) = \frac{1}{n+1}f(X) + \frac{n}{n+1}f(A_2).$$

Eliminating $f(Y)$ gives the desired result.

We will now use induction to show that $P_2 \Rightarrow P_n$ for all $n \geq 2$, the case $n = 2$ being immediate. Let

$$G = \frac{1}{n+1}\sum_{i=1}^{n+1} A_i \quad \text{and} \quad G' = \frac{1}{n}\sum_{i=1}^{n} A_i.$$

Hence,

$$G = \frac{n}{n+1}G' + \frac{1}{n+1}A_{n+1}.$$

Therefore,

$$f(G) = \frac{n}{n+1}f(G') + \frac{1}{n+1}f(A_{n+1}) \text{ (by (1))}$$

$$= \frac{n}{n+1}\left(\frac{1}{n}\sum_{i=1}^{n} f(A_i)\right) + \frac{1}{n+1} f(A_{n+1}) \text{ (by induction)}$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1} f(A_i),$$

as desired.

To show that $P_n \Rightarrow P_2$, let $M = (A_1 + A_2)/2$. Then,

$$f\left(\frac{1}{n}\left(M + M + \sum_{i=3}^{n} A_i\right)\right) = f\left(\frac{1}{n}\left(A_1 + A_2 + \sum_{i=3}^{n} A_i\right)\right)$$

$$\frac{1}{n}\left(2f(M) + \sum_{i=3}^{n} f(A_i)\right) = \frac{1}{n}\left(f(A_1) + f(A_2) + \sum_{i=3}^{n} f(A_i)\right) \text{ (by } P_n\text{)},$$

so $f(M) = (f(A_1) + f(A_2))/2$ as we wished to show.

A1124. The answer is yes. Note that if $1/F_n < x \le 1/F_{n-1}$ with $n \ge 3$, then

$$0 < x - \frac{1}{F_n} \le \frac{1}{F_{n-1}} - \frac{1}{F_n} \le \frac{2}{F_n} - \frac{1}{F_n} = \frac{1}{F_n}.$$

For $y \le 1$, let $g(y)$ denote the unique positive integer m such that

$$\frac{1}{F_m} < y \le \frac{1}{F_{m-1}}.$$

The relation above shows that $g(x - 1/F_n) > n$. Now take $x_1 = 1$, $n_1 = 3$ and recursively define

$$x_{k+1} = x_k - \frac{1}{F_{n_k}} \quad \text{and} \quad n_{k+1} = g(x_{k+1}).$$

This gives

$$1 = \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_6} + \frac{1}{F_9} + \frac{1}{F_{11}} + \frac{1}{F_{21}} + \frac{1}{F_{23}} + \cdots.$$

Note that the analogous result holds for any a such that

$$0 < a \le \sum_{n=1}^{\infty} \frac{1}{F_n} = 3.35988\ldots.$$