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Problems and Solutions

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PROBLEMS

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Proposals

To be considered for publication, solutions should be received by March 1, 2022.

2126. *Proposed by M. V. Channakeshava, Bengaluru, India.*

A tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the x-axis and y-axis at the points A and B, respectively.

Find the minimum value of AB.

2127. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA and Roger Horn, Tampa, FL.*

Suppose that $A, B \in M_{n\times n}(\mathbb{C})$ such that $AB = A$ and $BA = B$. Show that

(a) A and B are idempotent and have the same null space.
(b) If $1 \leq \text{rank } A < n$, then there are infinitely many choices of B that satisfy the hypotheses.
(c) $A = B$ if and only if $A - I$ and $B - I$ have the same null space.

2128. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $0 < a < b < 1$ and $\epsilon > 0$ be given. Prove the existence of positive integers m and n such that $(1 - b^m)^n < \epsilon$ and $(1 - a^m)^n > 1 - \epsilon$.

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

2129. *Proposed by Vincent Coll and Daniel Conus, Lehigh University, Bethlehem, PA and Lee Whitt, San Diego, CA.*

Determine whether the following improper integrals are convergent or divergent.

(a) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{2^k}\right) dx$

(b) $\displaystyle\int_0^1 \exp\left(\sum_{k=0}^{\infty} x^{3^k}\right) dx$

2130. *Proposed by Florin Stanescu, Şerban Cioiculescu School, Găeşti, Romania.*

Given the acute triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Choose $P, R \in \overleftrightarrow{AB}$, $S, T \in \overleftrightarrow{BC}$, $Q, U \in \overleftrightarrow{AC}$ so that

$$D \in \overleftrightarrow{PQ},\ E \in \overleftrightarrow{RS},\ F \in \overleftrightarrow{TU} \text{ and } \overleftrightarrow{PQ} \parallel \overleftrightarrow{EF},\ \overleftrightarrow{RS} \parallel \overleftrightarrow{DF},\ \overleftrightarrow{TU} \parallel \overleftrightarrow{DE}.$$

Show that

$$\frac{PQ + RS - TU}{AB} + \frac{RS + TU - PQ}{BC} + \frac{TU + PQ - RS}{AC} = 2\sqrt{2}$$

if and only if the circumcenter of $\triangle ABC$ lies on the incircle of $\triangle ABC$.

Quickies

1113. *Proposed by Philippe Fondanaiche, Paris, France.*

A generic n-gon is a convex polygon in which no three diagonals meet at a point in the interior of the n-gon. Determine the total number of triangles lying in the interior of a generic n-gon all of whose sides lie on the diagonals or sides of the n-gon.

1114. *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*

Let F_k denote the kth Fibonacci number defined by initial values $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_{k+2} = F_{k+1} + F_k$ for $k \geq 0$. Find the value of the sum

$$\sum_{k=2}^{\infty} \arctan \frac{F_{k-1}}{F_k F_{k+1} + 1} \arctan \frac{F_{k+2}}{F_k F_{k+1} - 1}.$$

Solutions

Invariance of a ratio of sums of cotangents **October 2020**

2101. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore MD and Mark Kaplan, Towson University, Towson, MD.*

Recall that the Steiner inellipse of a triangle is the unique ellipse that is tangent to each side of the triangle at the midpoints of those sides. Consider the Steiner inellipse E_S of $\triangle ABC$ and another ellipse, E_A, passing through the centroid G of $\triangle ABC$ and tangent to $\overleftrightarrow{AB}$ at B and to $\overleftrightarrow{AC}$ at C. If E_S and E_A meet at M and N, let $\angle MAN = \alpha$. Construct ellipses E_B and E_C, introduce their points of intersection with E_S, and define angles β and γ in an analogous way. Prove that

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{\cot A + \cot B + \cot C} = \frac{11}{3\sqrt{5}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland.
We first consider the equilateral triangle with vertices

$$A = (16, 0),\ B = (-8, 8\sqrt{3}),\ \text{ and }\ C = (-8, -8\sqrt{3}),$$

whose centroid is the origin. In this case, E_S is the circle whose equation is $x^2 + y^2 = 8^2$ and E_A is the circle whose equation is $(x + 16)^2 + y^2 = 16^2$. Solving this system of equations we find

$$M = (-2, 2\sqrt{15})\ \text{ and }\ N = (-2, -2\sqrt{15}).$$

Let $\angle(\overrightarrow{u}, \overrightarrow{v})$ denote the angle between the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$. Then

$$A = \angle\left((-24, 8\sqrt{3}), (-24, -8\sqrt{3})\right)\ \text{ and }\ \alpha = \angle\left((-18, 2\sqrt{15}), (-18, -2\sqrt{15})\right).$$

Rotating the vectors above 120° and 240° counter-clockwise gives

$$B = \angle\left((0, -16\sqrt{3}), (24, -8\sqrt{3})\right),$$

$$\beta = \angle\left((9 - 3\sqrt{5}, -9\sqrt{3} - \sqrt{15}), (9 + 3\sqrt{5}, -9\sqrt{3} + \sqrt{15})\right),$$

$$C = \angle\left((24, 8\sqrt{3}), (0, 16\sqrt{3})\right),\ \text{ and}$$

$$\gamma = \angle\left((9 + 3\sqrt{5}, 9\sqrt{3} - \sqrt{15})), (9 - 3\sqrt{5}, 9\sqrt{3} + \sqrt{15})\right).$$

Now let $\triangle A'B'C'$ be any non-degenerate triangle whose centroid is at the origin. There is an invertible linear map $f(x, y) = (ax + by, cx + dy)$ such that $\triangle A'B'C' = f(\triangle ABC)$. This linear mapping preserves the centroid, all midpoints, all tangencies, and it maps lines to lines and circles to ellipses. It remains to analyze how this linear mapping transforms the six numbers $\cot A$, $\cot B$, $\cot C$, $\cot\alpha$, $\cot\beta$, and $\cot\gamma$ to $\cot A'$, $\cot B'$, $\cot C'$, $\cot\alpha'$, $\cot\beta'$, and $\cot\gamma'$.

We will use the fact if $\phi = \angle((u_1, u_2), (v_1, v_2))$, then

$$\cot\phi = \frac{u_1v_1 + u_2v_2}{u_1v_2 - u_2v_1}$$

by the difference formula for cotangent.

Now

$$A' = \angle\left(f(-24, 8\sqrt{3}), f(-24, -8\sqrt{3})\right),$$

$$B' = \angle\left(f(0, -16\sqrt{3}), f(24, -8\sqrt{3})\right),\ \text{ and}$$

$$C' = \angle\left(f(24, 8\sqrt{3}), f(0, 16\sqrt{3})\right).$$

This gives

$$\cot A' = \frac{3a^2 - b^2 + 3c^2 - d^2}{2\sqrt{3}(ad - bc)}$$

$$\cot B' = \frac{b^2 - \sqrt{3}ab + d^2 - \sqrt{3}cd}{\sqrt{3}(ad - bc)}$$

$$\cot C' = \frac{b^2 + \sqrt{3}ab + d^2 + \sqrt{3}cd}{\sqrt{3}(ad - bc)}.$$

Therefore,

$$\cot A' + \cot B' + \cot C' = \frac{\sqrt{3}\left(a^2 + b^2 + c^2 + d^2\right)}{2(ad - bc)}.$$

A similar calculation yields

$$\cot \alpha' + \cot \beta' + \cot \gamma' = \frac{11\left(a^2 + b^2 + c^2 + d^2\right)}{2\sqrt{15}(ad - bc)}.$$

Finally,

$$\frac{\cot \alpha' + \cot \beta' + \cot \gamma'}{\cot A' + \cot B' + \cot C'} = \frac{11}{3\sqrt{5}}$$

as desired.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia) and the proposers. There were two incomplete or incorrect solutions.

Trigonometric identities for the heptagonal triangle **October 2020**

2102. *Proposed by Donald Jay Moore, Wichita, KS.*

Let $\alpha = \pi/7$, $\beta = 2\pi/7$, and $\gamma = 4\pi/7$. Prove the following trigonometric identities.

$$\frac{\cos^2 \alpha}{\cos^2 \beta} + \frac{\cos^2 \beta}{\cos^2 \gamma} + \frac{\cos^2 \gamma}{\cos^2 \alpha} = 10,$$

$$\frac{\sin^2 \alpha}{\sin^2 \beta} + \frac{\sin^2 \beta}{\sin^2 \gamma} + \frac{\sin^2 \gamma}{\sin^2 \alpha} = 6,$$

$$\frac{\tan^2 \alpha}{\tan^2 \beta} + \frac{\tan^2 \beta}{\tan^2 \gamma} + \frac{\tan^2 \gamma}{\tan^2 \alpha} = 83.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
Denote the trigonometric expressions by $\mathcal{C}, \mathcal{S}, \mathcal{T}$, respectively. The expansion

$$\sin(7t) = \sin t\left(64\cos^6 t - 80\cos^4 t + 24\cos^2 t - 1\right)$$

yields the key polynomial as follows. When $t = \alpha$ or $t = \beta$ or $t = \gamma$, then $\sin(7t) = 0$ but $\sin t \neq 0$. Hence the cubic polynomial

$$p(x) = 64x^3 - 80x^2 + 24x - 1$$

has the three zeros $a = \cos^2\alpha$, $b = \cos^2\beta$, $c = \cos^2\gamma$. Since

$$p(x) = 64(x-a)(x-b)(x-c),$$

we have values for the three elementary symmetric polynomials:

$$a+b+c = \frac{5}{4}, \quad ab+bc+ca = \frac{3}{8}, \quad abc = \frac{1}{64}.$$

We use the double angle formula for sine as follows:

$$\frac{\sin^2 t}{\sin^2 2t} = \frac{\sin^2 t}{4\sin^2 t\cos^2 t} = \frac{1}{4\cos^2 t}.$$

Hence, since $\sin^2 2\gamma = \sin^2\alpha$,

$$\mathcal{S} = \frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\gamma} + \frac{\sin^2\gamma}{\sin^2\alpha} = \frac{1}{4a} + \frac{1}{4b} + \frac{1}{4c} = \frac{bc+ca+ab}{4abc} = \frac{3/8}{4/64} = 6.$$

We use the double angle formula for cosine as follows:

$$\frac{\cos^2 t}{\cos^2 2t} = \frac{\cos^2 t}{(2\cos^2 t - 1)^2}.$$

Hence, since $\cos^2 2\gamma = \cos^2\alpha$,

$$\mathcal{C} = \frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\gamma} + \frac{\cos^2\gamma}{\cos^2\alpha} = \frac{a}{(2a-1)^2} + \frac{b}{(2b-1)^2} + \frac{c}{(2c-1)^2}.$$

Substituting $x = (y+1)/2$ into the polynomial $p(x)$ yields

$$q(y) = 8y^3 + 4y^2 - 4y - 1.$$

Since $y = 2x-1$, the zeros of $q(y)$ are $a' = 2a-1$, $b' = 2b-1$, $c' = 2c-1$ and the elementary symmetric polynomial expressions are

$$a'+b'+c' = -\frac{1}{2}, \quad a'b'+b'c'+c'a' = -\frac{1}{2}, \quad a'b'c' = \frac{1}{8}.$$

Hence,

$$\begin{aligned}\mathcal{C} &= \frac{a'+1}{2a'^2} + \frac{b'+1}{2b'^2} + \frac{c'+1}{2c'^2} = \frac{a'b'^2c'^2 + b'a'^2c'^2 + c'a'^2b'^2 + b'^2c'^2 + a'^1c'^2 + a'^2b'^2}{2(a'b'c')^2} \\ &= \frac{(a'b'c')(a'b'+b'c'+c'a') + (a'b'+b'c'+c'a')^2 - 2(a'b'c')(a'+b'+c')}{2(a'b'c')^2} \\ &= \frac{-1/16 + 1/4 + 1/8}{2/64} = 10.\end{aligned}$$

For the third identity, we use both double angle formulas:

$$\frac{\tan^2 t}{\tan^2 2t} = \frac{\sin^2 t\ \cos^2 2t}{\cos^2 t\ \sin^2 2t} = \frac{(2\cos^2 t - 1)^2}{4\cos^4 t}$$

Thus, since $\tan^2 2\gamma = \tan^2\alpha$,

$$\mathcal{T} = \frac{\tan^2\alpha}{\tan^2\beta} + \frac{\tan^2\beta}{\tan^2\gamma} + \frac{\tan^2\gamma}{\tan^2\alpha} = \left(\frac{2a-1}{2a}\right)^2 + \left(\frac{2b-1}{2b}\right)^2 + \left(\frac{2c-1}{2c}\right)^2.$$

Substituting $x = 1/(2(1-z))$ into the polynomial $p(x)$ and clearing fractions yields

$$r(z) = 8(z^3 + 9z^2 - z - 1).$$

Since $z = (2x-1)/(2x)$, the zeros of $r(z)$ are

$$a' = \frac{2a-1}{2a}, \quad b' = \frac{2b-1}{b}, \quad c' = \frac{2c-1}{c}$$

and the elementary symmetric polynomial expressions are

$$a' + b' + c' = -9, \quad a'b' + b'c' + c'a' = -1, \quad a'b'c' = 1.$$

Hence,

$$\mathcal{T} = a'^2 + b'^2 + c'^2 = (a' + b' + c')^2 - 2(a'b' + b'c' + c'a') = 9^2 - 2(-1) = 83.$$

Also solved by Michel Bataille (France), Anthony J. Bevelacqua, Brian Bradie, Robert Calcaterra, Hongwei Chen, John Christopher, Robert Doucette, Habib Y. Far, J. Chris Fisher, Dmitry Fleischman, Michael Goldenberg & Mark Kaplan, Russell Gordon, Walther Janous (Austria), Kee-Wai Lau (Hong Kong), James Magliano, Ivan Retamoso, Volkhard Schindler (Germany), Randy Schwartz, Allen J.Schwenk, Albert Stadler (Switzerland), Seán M. Stewart (Australia), Enrique Treviño, Michael Vowe (Switzerland), Edward White & Roberta White, Lienhard Wimmer (Germany), and the proposer. There were two incomplete or incorrect solutions.

How many tickets to buy to guarantee three out of four? October 2020

2103. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.*

In a soccer game there are three possible outcomes: a win for the home team (denoted 1), a draw (denoted X), or a win for the visiting team (denoted 2). If there are n games, betting slips are printed for all 3^n possible outcomes. For four games, what is the minimum number of slips you must purchase to guarantee that at least three of the outcomes are correct on at least one of your slips?

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

The answer is nine.

First, we prove that it is impossible to guarantee at least three correct outcomes with fewer than nine slips.

Let T be the set of all possible outcomes, i.e., all 4-tuples of 1, X, and 2. There are $3^4 = 81$ such 4-tuples. In that set, we define the Hamming distance d as the number of places in which two tuples differ. For example, $d(1X21, 2X12) = 3$ because 1X21 and 2X12 differ in three places, namely the first, third and fourth places. The Hamming distance satisfies the usual axioms for a metric, and we can define balls in T in the usual way, i.e., a ball with center $c \in T$ and radius $r \in \mathbb{R}$ is

$$B_r(c) = \{t \in T \mid d(t, c) \le r\}.$$

Given a tuple $c \in T$, the set of tuples that coincide with c in at least three places consists of those that differ from c in no more than one place. In other words, this set is $B_1(c)$. Note that $B_1(c)$ contains exactly 9 elements: the center c, the two tuples that differ from c exactly in the first element, the two that differ in the second, the two that differ in the third, and the two that differ in the fourth.

In order to ensure that our slips $c_1, c_2, \ldots, c_n$ contain at least three correct entries, the balls $B_1(c_i)$, $i = 1, 2, \ldots, n$ must cover T, i.e.,

$$T = \bigcup_{i=1}^{n} B_1(c_i).$$

Since $|B_1(c)| = 9$ and $|T| = 81$, we will need at least $81/9 = 9$ slips.

Next, we will prove that nine slips suffice. That can be accomplished by exhibiting nine 4-tuples $c_1, \ldots, c_9$ such that $B_i(c_1), \ldots, B_i(c_9)$ cover T, i.e., such that every element in T has a Hamming distance of at most 1 from at least one of the c_i. The following 4-tuples satisfy the condition:

1111 1XXX 1222 X1X2 XX21 X21X 2X12 212X 22X1

One (somewhat tedious) way to check it is to verify that each of the 81 elements in T differ from at least one of these tuples in no more one place.

A slightly easier way to verify the assertion is to observe that these tuples differ from each other in exactly three places, so the Hamming distance between any two of them is 3. Because of the triangle inequality, it is impossible for balls of radius 1 centered on the c_i to overlap. Therefore the total number of elements contained in the union of these balls is $9 \cdot 9 = 81$, so the union must be all of T.

This completes the proof.

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Eagle Problem Solvers, Fresno State Problem Solving Group, Dan Hletko, Rob Pratt, Allen J. Schwenk, and the proposer. There were seven incomplete or incorrect solutions.

Vector spaces as unions of proper subspaces **October 2020**

2104. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

It is well known that no vector space can be written as the union of two proper subspaces. For which m does there exist a vector space V that can be written as a union of m proper subspaces with this collection of subspaces being minimal in the sense that no union of a proper subcollection is equal to V?

Solution by Paul Budney, Sunderland, MA.

Such a decomposition exists for any $m > 2$.

Let $V = \mathbb{F}_2^n$, where $\mathbb{F}_2$ is the field with two elements. Let

$$V_i = \{(x_1, \ldots, x_n) \in V \,|\, x_i = 0\}$$

for $1 \le i \le n$ and let

$$W = \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}.$$

Clearly W and the V_i are proper subspaces of V. Since $(1, 1, \ldots, 1)$ is the only vector not in $V_1 \cup V_2 \cup \ldots \cup V_n$,

$$W \cup V_1 \cup V_2 \cup \ldots \cup V_n = V.$$

Deleting W from this union excludes $(1, 1, \ldots, 1)$. Deleting V_i from this union excludes $(1, \ldots, 1, 0, 1, \ldots, 1)$, with 0 for the ith component and 1's elsewhere. Thus, there is no proper subcollection of these subspaces whose union is V. There are

$n+1$ subspaces, and since $n \geq 2$ is arbitrary, the desired decomposition exists for any $m > 2$.

Also solved by Anthony Bevelacqua, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Doucette, Eugene Herman, and the proposer. There was one incomplete or incorrect solution.

An asymptotic formula for a definite integral **October 2020**

2105. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bîrlad, Romania.*

Let $f : [0,1] \to \mathbb{R}$ be a function that is k times differentiable on $[0,1]$, with the kth derivative integrable on $[0,1]$ and (left) continuous at 1. For integers $i \geq 1$ and $j \geq 0$ let

$$\sigma_j^{(i)} = \sum_{j_1+j_2\cdots+j_i=j} 1^{j_1}2^{j_2}\cdots i^{j_i},$$

where the sum is extended over all i-tuples $(j_1,\ldots,j_i)$ of nonnegative integers that sum to j. Thus, for example, $\sigma_0^{(i)} = 1$, and $\sigma_1^{(i)} = 1+2+\cdots+i = i(i+1)/2$ for all $i \geq 1$. Also, for $0 \leq j \leq k$ let

$$a_j = \sigma_j^{(1)} f(1) + \sigma_{j-1}^{(2)} f'(1) + \cdots + \sigma_1^{(j)} f^{(j-1)}(1) + \sigma_0^{(j+1)} f^{(j)}(1).$$

Prove that

$$\int_0^1 x^n f(x)dx = \frac{a_0}{n} - \frac{a_1}{n^2} + \cdots + (-1)^k \frac{a_k}{n^{k+1}} + o\left(\frac{1}{n^{k+1}}\right),$$

for $n \to \infty$. As usual, we denote by $f^{(s)}$ the sth derivative of f (with $f^{(0)} = f$), and by $o(x_n)$ a sequence (y_n) with the property that $\lim_{n\to\infty} y_n/x_n = 0$.

Solution by Michel Bataille, Rouen, France.
For $x \in [0,1]$, let $f_0(x) = f(x)$ and

$$f_j(x) = \frac{d}{dx}\left(xf_{j-1}(x)\right), 1 \leq j \leq k.$$

An easy induction shows that for $0 \leq j \leq k$, the function f_j is a linear combination of the functions $f(x), xf'(x), \ldots, x^j f^{(j)}(x)$. It follows that $f_0, f_1, \ldots, f_{k-1}$ are differentiable on $[0,1]$ and that f_k is integrable on $[0,1]$ and continuous at 1.

Integrating by parts, we obtain the following recursion that holds for $1 \leq j \leq k-1$:

$$\begin{aligned}\int_0^1 x^n f_{j-1}(x)\,dx &= \left[\frac{x^n}{n}\cdot(xf_{j-1}(x))\right]_0^1 - \frac{1}{n}\int_0^1 x^n f_j(x)\,dx \\ &= \frac{f_{j-1}(1)}{n} - \frac{1}{n}\int_0^1 x^n f_j(x)\,dx.\end{aligned}$$

With the help of this recursion, we are readily led to

$$\int_0^1 x^n f(x)\,dx = \int_0^1 x^n f_0(x)\,dx$$

$$= \sum_{j=0}^{k-1} (-1)^j \frac{f_j(1)}{n^{j+1}} + \frac{(-1)^k}{n^k} \int_0^1 x^n f_k(x)\, dx.$$

Now, if $g : [0, 1] \to \mathbb{R}$ is integrable on $[0, 1]$ and continuous at 1, then

$$\lim_{n\to\infty} n \cdot \int_0^1 x^n g(x)\, dx = g(1)$$

(Paulo Ney de Souza, Jorge-Nuno Silva, *Berkeley Problems in Mathematics*, Springer, 2004, Problem 1.2.13). With $g = f_k$, this yields

$$\int_0^1 x^n f_k(x)\, dx = \frac{f_k(1)}{n} + o\left(\frac{1}{n}\right)$$

and therefore

$$\begin{aligned}\int_0^1 x^n f(x)\, dx &= \sum_{j=0}^{k-1} (-1)^j \frac{f_j(1)}{n^{j+1}} + \frac{(-1)^k}{n^k} \left(\frac{f_k(1)}{n} + o\left(\frac{1}{n}\right)\right) \\ &= \sum_{j=0}^{k} (-1)^j \frac{f_j(1)}{n^{j+1}} + o\left(\frac{1}{n^{k+1}}\right).\end{aligned}$$

Comparing this with the statement of the problem, it remains to prove that $a_j = f_j(1)$ for $0 \le j \le k$. Clearly, it is sufficient to prove that for $x \in [0, 1]$

$$f_j(x) = \sum_{i=0}^{j} \sigma_{j-i}^{(i+1)} x^i f^{(i)}(x). \qquad (E_j)$$

We use induction. Since $f_0(x) = f(x) = 1 \cdot x^0 f^{(0)}(x)$, (E_0) holds. Before addressing the induction step, we establish two results about the numbers $\sigma_j^{(i)}$. The first result is

$$\sigma_j^{(i+1)} = \sum_{r=0}^{j} (1+i)^r \sigma_{j-r}^{(i)}. \qquad (1)$$

Proof. When $j_1 + \cdots + j_i + j_{i+1} = j$, then j_{i+1} can take the values $0, 1, \ldots, j$. It follows that

$$\begin{aligned}\sigma_j^{(i+1)} &= \sum_{j_1+\cdots+j_{i+1}=j} 1^{j_1} 2^{j_2} \cdots i^{j_i} (i+1)^{j_{i+1}} \\ &= \sum_{r=0}^{j} (1+i)^r \sum_{j_1+\cdots+j_i=j-r} 1^{j_1} 2^{j_2} \cdots i^{j_i} \\ &= \sum_{r=0}^{j} (1+i)^r \sigma_{j-r}^{(i)}.\end{aligned}$$

The second result is

$$\sigma_{j+1}^{(i+1)} = \sigma_{j+1}^{(i)} + (1+i)\sigma_j^{(i+1)}. \qquad (2)$$

Proof. Applying (1),

$$\begin{aligned}\sigma_{j+1}^{(i+1)} &= \sum_{r=0}^{j+1}(1+i)^r\sigma_{j+1-r}^{(i)}\\ &= \sigma_{j+1}^{(i)} + (1+i)\sum_{r=1}^{j+1}(1+i)^{r-1}\sigma_{j-(r-1)}^{(i)}\\ &= \sigma_{j+1}^{(i)} + (1+i)\sum_{r=0}^{j}(1+i)^r\sigma_{j-r}^{(i)}\end{aligned}$$

and applying (1) again we conclude that $\sigma_{j+1}^{(i+1)} = \sigma_{j+1}^{(i)} + (1+i)\sigma_j^{(i+1)}$.

Now, assume that (E_j) holds for some integer j such that $0 \leq j \leq k-1$. Then, we calculate

$$\begin{aligned}f_{j+1}(x) &= \frac{d}{dx}\left[\sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}x^{i+1}f^{(i)}(x)\right]\\ &= \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}(i+1)x^i f^{(i)}(x) + \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}x^{i+1}f^{(i+1)}(x)\\ &= \sum_{i=0}^{j}\sigma_{j-i}^{(i+1)}(i+1)x^i f^{(i)}(x) + \sum_{i=1}^{j+1}\sigma_{j-i+1}^{(i)}x^i f^{(i)}(x)\\ &= \sigma_j^{(1)}f(x) + \sum_{i=1}^{j}\Big([\sigma_{j-i+1}^{(i)} + (i+1)\sigma_{j-i}^{(i+1)}]x^i f^{(i)}(x)\Big) + \sigma_0^{(j+1)}x^{j+1}f^{(j+1)}(x).\end{aligned}$$

Using (2) and $\sigma_j^{(1)} = \sigma_{j+1}^{(1)} = 1 = \sigma_0^{(j+1)} = \sigma_0^{(j+2)}$, we see that

$$f_{j+1}(x) = \sum_{i=0}^{j+1}\sigma_{j+1-i}^{(i+1)}x^i f^{(i)}(x)$$

so that (E_{j+1}) holds. This completes the induction step and the proof.

Note. The number $\sigma_j^{(i)}$ is the Stirling number of the second kind $S(i+j, i) = \left\{{i+j \atop i}\right\}$ (see L. Comtet, *Advanced Combinatorics*, Reidel, 1974, Theorem D p. 207).

Also solved by Albert Stadler (Switzerland) and the proposer.

Answers

Solutions to the Quickies from page 309.

A1113. Consider the union of the endpoints of the sides or diagonals of the polygon that contain the sides of the interior triangle. There can be 3,4,5, or 6 such points. Order those points in a clockwise direction around the polygon: $P_1, P_2, \ldots$.

- With three points, the sides of the triangle must be $\{P_1P_2, P_1P_3, P_2P_3\}$. There are $\binom{n}{3}$ of these triangles.

- With four points, there are four possibilities for the sides of the triangle: $\{\{P_i P_{i+2}, P_{i+2} P_{i+1}, P_{i+1} P_{i+3}\}\}$, where the subscripts are taken modulo 4. There are $4\binom{n}{4}$ of these triangles.
- With five points, there are five possibilities for the sides of the triangle: $\{\{P_i P_{i+2}, P_{i+2} P_{i+4}, P_{i+1} P_{i+3}\}\}$, where the subscripts are taken modulo 5. There are $5\binom{n}{5}$ of these triangles.
- With six points, the sides of the triangle must be $\{P_1 P_4, P_2 P_5, P_3 P_6\}$. There are $\binom{n}{6}$ of these triangles.

Therefore, the total number of triangles is

$$\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}.$$

A1114. Let

$$S_n = \sum_{k=2}^{n} \arctan \frac{F_{k-1}}{F_k F_{k+1} + 1} \arctan \frac{F_{k+2}}{F_k F_{k+1} - 1}.$$

By the recurrence relation for the Fibonacci numbers, we have

$$S_n = \sum_{k=2}^{n} \arctan \frac{F_{k+1} - F_k}{F_k F_{k+1} + 1} \arctan \frac{F_k + F_{k+1}}{F_k F_{k+1} - 1}.$$

Since,

$$\arctan \frac{y - x}{xy + 1} = \arctan \frac{1}{x} - \arctan \frac{1}{y} \quad \text{and} \quad \arctan \frac{y + x}{xy - 1} = \arctan \frac{1}{x} + \arctan \frac{1}{y}$$

for $xy > 1$, our sum becomes

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \left(\arctan \frac{1}{F_k} - \arctan \frac{1}{F_{k+1}}\right)\left(\arctan \frac{1}{F_k} + \arctan \frac{1}{F_{k+1}}\right) \\ &= \sum_{k=2}^{n} \left(\arctan \frac{1}{F_k}\right)^2 - \left(\arctan \frac{1}{F_{k+1}}\right)^2 \\ &= \left(\arctan \frac{1}{F_2}\right)^2 - \left(\arctan \frac{1}{F_{n+1}}\right)^2, \end{aligned}$$

since the last sum telescopes. Hence, the sum of the series is

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \left(\left(\arctan \frac{1}{F_2}\right)^2 - \left(\arctan \frac{1}{F_{n+1}}\right)^2\right) \\ &= \left(\arctan \frac{1}{F_2}\right)^2 = \frac{\pi^2}{16}. \end{aligned}$$